Y. MIKAMI

MATHEMATICAL PAPERS FROM THE FAR EAST

B. G. Teubners Sammlung von Lehrbüchern auf dem Gebiete der Mathematischen Wissenschaften mit Einschluß ihrer Anwendungen.

gr. 8. Geb.

Diese Sammlung bietet in einzelnen in sich abgeschlossenen Werken zusammenfassende Darstellungen der wichtigsten Abschnitte der mathematischen Wissenschaften und deren Anwendungen. Im einzelnen wollen diese Werke in ihrer ausführlichen, neben der rein wissenschaftlichen auch pädagogische Momente berücksichtigenden Darstellung die Möglichkeit zu selbständigem und von umfangreichen Quellenstudien unabhängigem Eindringen in die verschiedenen Disziplinen geben; in ihrer Gesamtheit aber sollen sie durch ihre eingehenden literarischen und historischen Nachweise ein genaues Bild von der modernen Entwicklung ... wärtigen Stande der ... wendungen darbieten.

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☞ Nähere Angaben über obige Werke befinden sich in meinem mathematischen Katalog, den ich zu verlangen bitte. Verlagsanerbieten für die Sammlung werden mir jederzeit willkommen sein.

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COMPILED AND EDITED

BY

YOSHIO MIKAMI

WITH 15 FIGURES IN THE TEXT

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PREFACE.

In the present work I have endeavoured to represent the studies made by Japanese mathematicians of our age. The papers have most of them been reproduced in more or less condensed forms, and they consist of those that were originally given in our own tongue, except one or two that come forth in connection with others. I don't mean to have brought all choice productions of the Japanese mind in this one volume, nor I pretend to have exhausted those writings that could be taken for representatives. I shall be satisfied, if I could give to my readers any image or aspect of the subjects our Japanese scholars are fond of pursuing and taking their tastes in. As to the results brought out in this meagre composition, I, with any other Japanese, persist no way on their originality with us Japanese. We well persist however that these are all original studies crowning the efforts of the respective Japanese authors, — not borrowed from any outer world. On that account I desire to believe that these should no doubt be worth raising some interest from the students of things Japanese — historical or social; which is really the why I have taken up the pains of compiling and editing the present collection.

It is not quite half a century since the Occidental style of learnings brought Japan under the overflow of its sway. But we Japanese have achieved in that short interval of time what we now possess. For the contents of this book and like productions we Japanese do not but feel the growth of a thanking heart towards the capable and prominent leadership of our European predecessors.

It is true we Oriental people had our own civilization even from before our contacts with the Western world. Our forefathers and their predecessors the Chinese did much indeed in the advancement of their mathematical knowledge from time unmemorial. I therefore in way of introduction give a short notice on the progress of mathematics in

China and Japan, which, I hope, may perhaps be served for comparison with what I have to represent in the forelying volume, that embodies our worthy contemporaries' productions.

On the eve of finishing in my work I desire to take occasion of expressing my feeling of respects to the respective authors, whose publications have enriched my collection. And my thanks are especially due to the publisher who has kindly undertaken to set the present work before public.

Ohara in Kazusa, September 26, 1907.

Yoshio Mikami.

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A SHORT NOTICE ON THE MATHEMATICS IN THE FAR EAST IN ITS DEVELOPMENT.

By way of an introduction to the present work we shall here give a short notice on the development of mathematics in China and Japan.

There are two of oldest works on mathematics that have been left by the ancient Chinese and that still remain preserved to our own day. The one of these is the Chou-pei and the other the Chiu-chang Suan-shu.

The Chou-pei is a calendrical work in which there is recorded a dialogue that had happened between the wise prince and sage Chou-Kong (d. 1105) and Shang Kao. It is not known who was the author of this book, or at what date it was composed of. Nor it remains in its original form; it has been preserved through the hands of successive commentators. Besides the latter half of it seems in all probability to be an addition made in a subsequent age. But in spite of all this the Chou-pei reveals the state of mathematical attainments, at least in parts, arrived at by the Chinese as early as the 12$^{th.}$ century B. C. As it points out, the geometrical theorem, so important in application, attributed to Pythagoras, was known to the Chinese of that remote antiquity, although not in its geometrically rigorous form. It was even applied to calendrical matters. The knowledge of ratio and proportion was of course required in such an application, and the extraction of a square root must have been carried out.

There are two sorts of the Chiu-chang, one attributed to Hoang-Ti, and the other bearing the title of the Chiu-chang Suan-shu, or Arithmetical Rules in Nine Sections. A work called the Hoang-Ti Chiu-chang prevailed some time at the close of Sung Dynasty, but we can put no great importance on it, being evidently a composition in a later age. The term Chiu-chang was employed, because arithmetical rules or rather questions were classified in nine sections. According to tradition these sections had descended from the time of Hoang-Ti, who reigned in the 27$^{th.}$ century B. C. At Chou-Kong's time a pre-

viously existing work was revised or a new one was specially composed of by his orders, and that bore the title of the Chiu-chang. We know however nothing of what a kind of work the Chiu-chang of Chou-Kong had been. We only know that the prince had wisely encouraged the study of arithmetic.

The Chiu-chang Suan-shu, that descends to us through various editions, was written by Chang T'sang, who held the office of chief minister from 176 to 162 B. C. in the reign of a Han emperor, and who died in 152 B. C. in his old age. Chang's life-time was just subsequent to the age of the disastrous fire when books were burnt by Shih Hoang-Ti of Ch'in in 213 B. C. But he searched after the remnants of old manuscripts, and having found a copy on the Chiu-chang, he composed the Chiu-chang Suan-shu. Although the work is said to base on a preceding document, yet it differed in various points, and it sought to newer way of phraseology, as the commentator Liu Hui affirms in his preface. It was revised in the next century by Ching Ch'ou-ch'ang.

The abucus arithmetic appears to have prevailed in every nation of antiquity. But where one can expect to see an abucus more convenient than employed in old China? Where one can meet with an abucus such that the Chinese kind has caused so extravagant a progress of mathematics in later years? The employment of that kind of abucus called the suan-ch'ou, or what have been called the sangis in modern Japan, — terms that can be rendered by *calculating pieces*, — had been undoubtedly practiced in China from time unmemorial.

The Chiu-chang Suan-shu appeared in the beginning of Han, as we observed. In the work there remains no stage devoted to the description of any instrument for calculation, but the explanations found there leave little room to be doubted as to the employment of the calculating pieces. That these pieces had been employed during the Han administration is mentioned by Pan Ku, who wrote the History of the Former Han Dynasty in the 1st century of Christian era. The pieces consisted of old of bamboo sticks not so short and convenient, as practiced in subsequent ages.

The use of abucus was by no means characteristic to the Chinese, but the very sort of the abucus employed in China has caused the progress of mathematics strike a way peculiar and unique to them. Buddhism was introduced and the Hindoo influence swept over the whole empire, in so much that all modes of learning went largely affected in idea and form. In the domain of mathematics were however

the Chinese free to follow their own direction of development — thanks to the use of the calculating pieces! The same abucus was also destined to work on the Japanese mind to establish a proper construction of mathematical attainments.

The methods treated in the Chiu-chang Suan-shu that are to be ascribed to the sangis cover the four rules and up to the extractions of the square and cube roots.

The calculating pieces consist of two sorts, which are distinguished by different colours, — red and black, — representing the positive and negative, or rather additive and subtractive numbers. This distinction was not stated in Chang's work, where however the terms positive and negative were made use of. The fang-ch'êng process, one of the nine sections, had for its object to solve a linear system of simultaneous equations, which was most likely arranged on the sangi board by means of the positive and negative pieces. The algebraical symbols in modern China and Japan have undoubtedly come from the way in which such an arrangement was to be recorded.

It is very remarkable that the Chiu-chang Suan-shu contains a problem, which was afterwards also given in Brahmagupta.

Of works next old to the two just described may be mentioned the treatises of Sun-tsŭ, Hsia-hou Yang, Chang Chiu-chien, Liu Hui, and others. These were not of course the whole of publications at the age we are concerned of; for the works of Hsü Shang and Tu Chung and the San-têng-shu, etc., are among the lost. But we have no means of pursuing the progress of the science in the After Han Period but from these extant works.

Sun-tsŭ's work is a treatise elaborated orderly in a high degree. The beautiful process of the t'ai-yen ch'iu-yi-shu, that was completed, through the hand of Yi-hsing of T'ang, by Ch'in Chiu-shang of Sung, was first met with in this book of Sun-tsŭ. The problem comes from a stage in the sacred book of Yi-ching.

Sun-tsŭ is sometimes taken for the illustrious tactician Sun Wu, who lived towards the end of the 6th. century B. C. But there is no testimony to believe it, according to some authorities he belonged to a period not earlier than the introduction of Buddhism.

The Hai-tao Suan-shu, or the Sea-islands Arithmetic, of Liu Hui of Wei, treats of measurements by the application of the relations that exist in a right-angled triangle. The book was originally appended to the same author's commentary on the Chiu-chang Suan-shu, that was published in 263 A. D.

In the beginning of the T'ang dynasty, that is, in the first half of the $7^{th.}$ century, there came out a great personage, whose memory could never be forgotten by one who concerns with the mathematical history. This learned man was Wang Hs'iao-t'ung, and his Ch'i-ku Suan-ching presented to his emperor remains extant to this very day. There is, in this work, applied an approximate way of solving cubic equations — harbinger of the algebraical methods of Ch'in Chiu-chang and Li Yeh that arose six centuries subsequently. Although the Chinese way of solving numerical equations were nothing but a natural extension from the same process that effects the extraction of the square and cube roots, as practiced on the sangi board, yet we cannot but admire Wang's possession of a firm footing in the progress of mathematics in China.

Buddhism was brought into China in 65 A. D., and the religion became predominating in the course of a few centuries, when things Indian exercised superiority over the proper civilization of ancient China. A writer records that Indian books were read in translations ten times more than classical works. In such an age how could mathematics alone remain uninfluenced? Chên Luan and other mathematicians were enthusiastic adherents to the religion. Some of Hindoo mathematical works, such as the Brahman Arithmetic and the Brahman Arithmetical Classic, it is stated in history, had been even translated into Chinese. In short Hindoo arithmetic had been studied in China.

But however great might have been the Hindoo influence in those times, the Chinese science, in so far as we can judge from the remnants presented before our eyes, remained unaffected. The traces of Hindoo mathematics have since all gone out of memory and lost for ever. As to Wang's study of the cubic equations, we can never take it for a foreign source of origin.

No value of π was adopted in the Chiu-chang Suan-shu more minute than 3 : 1, nor any calculation was tried for the determination of it. But the problem of circle measurement did not fail to come out soon afterwards and caught the attention of scholars, perhaps owing to the transplantation of idea from India. After various scholars had obtained various values, there appeared Liu Hui in the third century. He tried a measurement of the circle, and came with the value $\pi = 157/50$, result that, together with his calculation, has been embodied in a note to his edition of the Chiu-chang Suan-shu of 263. Liu took recourse to a regular hexagon inscribed in a circle in his trial. About the same time with Liu, Wang Fan, who belonged to a different kingdom, took for $\pi = 142/45$.

Two centuries after Liu Hui another trial for the same measurement was done by Tsu Ch'ung-chih (428—499), who both inscribed and circumscribed hexagons about a circle and proceeded therefrom in consideration. Tsu struck, although unknowingly, the same way as Archimedes had done hundreds of years previously. Tsu found that the value of π should lie between $3 \cdot 1415926$ and $3 \cdot 1415927$, from which he deduced the two fractional values 22/7 and 355/113 The first fraction was the same as one given by Archimedes, but as for the latter, where can we find it else? Neither the Grecians nor the Hindoos nor the Arabians had it. No page of history comes with it, until it was rediscovered quite lately in Holland. Tsu Ch'ung-chih had known of the fractional value upward of a thousand years before the Europeans. This discovery of Tsu remains recorded in the Records of Sui Dynasty — a work that was written by Wei Chih in the beginning of the 7th. century. Tsu's own writing, the Chui-shu, has been lost. Perhaps it should have been a calendrical work, in which the author's circle measurement was appended. Later scholars all agree in the belief that Tsu had undertaken some way or other of an expansion in infinite series, as the terms mentioned in Wei's History seem well to indicate.

The two fractions of Tsu were termed the rough value and the minute. But the first had become in course of time to be known by the latter name.

Tsu's results did no way close the consideration of the circle measurement. For Chang Yu-chin of Mongol Dynasty and other scholars have ever since repeatedly appeared in their way of attack. It must not be forgotten, however, that such a great mathematician like Ch'in Chiu-shang of Sung had to take the value $\pi = \sqrt{10}$, without making any test whatever. It is especially noteworthy when we reflect that the same value had been employed in Egypt, in Babylon, in India and in Arabia. Japanese mathematicians had also to adopt it in an early part of their history.

Li Ch'ung-fêng and other noted calendarmakers appeared in the beginning of T'ang. They were men skilled in the art of arithmetic. Li wrote by an imperial edict commentaries on various arithmetical treatises. Li was a contemporary of Wang.

Subsequent to the time of these scholars the Buddhist priest Yi-hsing made his appearance and constructed the T'ai-yen calendar, in which he employed the method of t'ai-yen-shu, a mode of indeterminate analysis.

After Yi-hsing mathematics appears to have undergone a gradual development through the long reigns of the dynasties of T'ang and Sung, an age of which however we are little meaned to be able to make any description.

We now come to the epoch where the closing days of the Sung Dynasty blends with the uprise of the Mongolian invaders. It was just at this juncture that the mathematics in China had attained the topmost mark of its development. We said above of Wang's solving numerical equations of the third degree. Such a solution was now extended to higher equations in general. At the same time there arose the algebraical way of treatment of expressions that contain an unknown quantity.

In 1247 Ch'in Chiu-shang of Sung wrote the Su-shu Chiu-chang, in which he explains in detail the solution of a numerical equation in any degree. Ch'in's method was an extension of the process of extracting square and cube roots as given in the Chiu-chang Suan-shu. It is almost the same as the method published by Horner in 1819. On this subject therefore the Chinese had forerun the Europeans by nearly six whole centuries. At Ch'in's time the calculation had been applied digit after digit. No abridgment of labour was as yet undertaken.

About the same time Li Yeh (1178—1265) wrote two treatises on algebra — the T'sê yüan Hai-ching of 1248 and the Yi-ku Yen-tuan of 1259; both of which treat of the so-called li-t'ien-yüan-shu, or the method of setting the heavenly element, by which is meant to represent the unknown quantity sought for evaluation. Li's works treat solely of the way how to get equations from the data put forward in the problems; in this way they widely deviate from Ch'in's explaining of the actual process of solving such equations without giving any words as to their construction.

Here we shall not discuss on the priority of the inventions of the two scholars Ch'in and Li. It will be sufficient, if we mention that the Chinese algebra and with it the process of solving numerical equations had arisen in consequence of the employment of the calculating pieces. On this account we never hesitate in ascribing the growth of algebra of the heavenly element to no foreign source of origin.

Half a century later Chu Shih-chieh wrote the Suan-hsiao Chi-mêng (1299), in which he treated the t'ien-yüan algebra, — a book of little value in the Chinese history, but which have displayed a heavy roll

on the development of mathematics in Japan. In 1303 his Szŭ-yüan Yü-chien followed. This latter is invaluable in history, because it embodies the highest attainment of the Chinese algebra, which was not carried on writing but by manipulating with the sangis. These pieces being arranged in the right and left, in the front and rear, it is capable to represent an expression in not more than four unknowns, from which a certain proceeding leads to eliminate the unknowns all but one, leaving the final expression in the heavenly element. Thus the algebraical treatment carried on the sangi-board became exceedingly elegant and at the same time interesting to the utmost.

Kuo Shou-ching (1231—1316), who was a great astronomer, applied the algebra of heavenly element in the establishment of his calendar system. Problems on spherical triangles were considered by him for the first time in China. His methods, or at least his problems, are said to have been borrowed from the Arabians.

We mentioned above of Hindoo influences. A short time before Yi-hsing's composition of his calendar there had been an Indian calendar translated. Although it has been since lost, we know, from the contents of the Records of T'ang Dynasty, that the written arithmetic was employed therein. Such arithmetic, however, did not continue very long to exercise power in China. It had soon disappeared from men's memory.

The intercourse with the Arabians began in the reign of the dynasty of T'ang, and they were even employed, during the Mongolian ascendency, in the astronomical board, so that nothing can deny of free introduction of Arabian sciences. It is rather surprising, therefore, that there hardly remains any trace of the written arithmetic being used in China.

The Mongolian yoke did not last long and the native dynasty of Ming succeeded. But the study of mathematics now ebbed. Although there had been those scholars like T'ang and K'u, they were little able to understand the essence of the systems founded by Ch'in and Li, Chu and Kuo.

On the other hand the reign of Ming Dynasty is remarkable for the rise of a custom of using a new instrument in daily-use calculations. It was the suan-pan, or the soroban of the Japanese. Although we meet with no explanation of this sort of abucus older than in the Suan-fa T'ung-tsung of 1592, it was no new invention. It is highly probable that the instrument, although in a different form, had existed from a remoter age.

The end of the Ming Dynasty saw the Christian religion introduced by the Jesuit priests. Matteo Ricci (1552—1610) and others were possessed of a tolerable knowledge in science and mathematics; so they took a good opportunity of their applications for the purpose of their mission. There soon appeared Hsü, Li and other scholars, who studied the Western sciences, and the missionaries translated, assisted by them, those works as Euclid, the T'ung-wen Suan-chih, and numerous others.

The missionaries were warmly received at court and they were situated in the astronomical board. Consequently a calendrical work was composed after the Occidental style. The mathematical science in China then presented an utterly changed condition.

The invasion of the Manchurians had left such a state of things untouched, and the Occidental learnings seemed to continue in their flourishment. But the Chinese, who always worship old times, came to take up older mode of sciences again in their favour, and the foreign influences remained without leaving as much effect as might seemed on the outset.

About the state of mathematics, that has come during the Manchurian yoke of present dynasty, we shall reserve it to another occasion of coming with the matter again.

Now we proceed to say something about the performances of the Japanese.

Chinese mathematics was once brought into Japan in those old times when the flood of civilization had inundated from the continent. Scholars studied their Chinese masters, but the study went down gradually, until there remained no trace of former ascendency.

Under Toyotomi's rule, it is stated, Mōri paid a visit to China in quest of mathematical knowledge. Be the story true or untrue, it is the same that mathematics had been brought for a second time from China, for the Suan-fa T'ung-tsung was the sole authority over Japanese mathematicians in the beginning of their uprise. The soroban arithmetic has been practiced in Japan from this epoch on.

The Jinkōki of Yoshida was published in 1627. It was an arithmetical treatise based on the above said work. It went through various editions.

The use of soroban did not bring the calculating pieces, that had been probably practiced from a more older date, into disuse. On the contrary the sangis were the more practiced when arithmetical studies had become the more popular. And especially so, when the Suan-hsiao

Chi-mêng had come into favour of Japanese mathematicians. This work was reprinted in Japan at different times with notes (1658, 1672 etc.). The effect done by this single book was amazingly great. The study of the method of the heavenly element, or as it must now be called, the tengen jutsu, soon arose.

In 1670 Sawaguchi wrote the Kokon Sampō-ki, in which he advanced a step upward of his Chinese master's performances, for he notices in the existence of a negative root and of more than one roots, in an equation, although he was little inclined to take them as authentic. He ascribed them to defects in the setting up of problems.

In the new edition of the Jinkōki, that was printed in 1641, Yoshida had given twelve questions and awaited his successors to work for answers. These problems were solved by several authors and their solutions were published in the Sanryō-roku (1653) of Yenami, the Yempō Shikanki (1657) of Hatsusaka, the Kaisan-ki (1659) of Yamada, the Ketsugi-shō (1661) of Iwamura, and others. These books were each provided with its own new problems, which were solved in turn in later publications. In this manner there arose the usage of solving problems given in a preceding work, when one had to compose a mathematical treatise. The problems in the Ketsugi-shō were answered in Nozawa's Dōkai-shō of 1664, and in Satō's Kongenki of 1666. Those in the Kongenki were the same which Sawaguchi had solved by means of the tengen-jutsu method in his Kokon Sampōki of 1670, a work in which there were given new problems also. These problems of Sawaguchi were solved by Seki Kōwa and were published in his Hatsubi Sampō four years later. One of the problems resulted indeed in an equation of the 1457$^{\text{th.}}$ degree, an equation that was not given in the said work except that it is stated to lead to such an equation. The same problem was considered afterwards by Miyagi, who published the whole of his calculation in his Wakan Sampō of 1695. See how tremendous, how tedious, such a consideration must have been!

The formulae in Seki's Hatsubi Sampō were explained by his pupil Takebe in 1685. The way followed in such a solution was termed the yendan process.

In the yendan process, it is true, symbols were employed to aid calculations. But they were employed in order to get at the final goal of setting the tengen expression for the sangi-board. Thus the process could not be taken for an algebraical proceeding genuine in its meaning; or at least our forefathers thought so. There arose consequently another method of tenzan jutsu, or independent algebra.

If the difference that exists between these two sorts of methods was so trivial as almost indiscernable to a modern mind, yet it had not been so with the ancients, who had to distinguish between them so great as heavens differ from earth. It appears, the tenzan solutions were carried entirely in writing. This establishment has made algebraical calculations very simple, very free; but the method remained long kept secret.

The tenzan jutsu was first given in a published work in 1766 in the Shūki Sampō, when the scholars in Seki's school were very displeased with such an act, but they had remained silent, because the author was a feudal lord in disguise under a feigned name. Even after that time the custom of keeping secret ever continued among mathematicians. The Shūki Sampō does not solve problems given in preceding publications. The usage came now to an end.

Seki Kōwa, who has been ever adored as the father of the Japanese mathematics, was born in 1642 near Yedo. He was most probably a self-formed man. Seki had already founded a written system of algebra. It was not derived evidently from Chinese works. But there arises naturally a question whether Seki had not been affected by the Occidental mode of learnings. While Seki was busily engaged in the construction of his mighty system, there was a certain Japanese, who pursued his studies in Holland. This Japanese was Petrus Hartsingius by name. About the same time there was a physician, Hatono Sōha, who had returned from abroad. It is uncertain if the two persons had been identical. In that case Seki could have studied from Hatono something, which however must have taken place in or after 1681, the date of Hatono's establishment in Ōsaka. Seki's invention appears however to have been effected in an earlier date. Besides Seki's notation had been derived from the Chinese scholars. Had he ever felt a shock of influence from an Occidental source, it should have certainly lain in his theorems, not in his system of algebra.

About the numerous theories invented by Seki we shall not dwell here to enter into detail. We shall content ourselves in stating that Seki had succeeded in striking an abridgment in the tengen method of solving equations, that he had made a discovery on the number and nature of the roots of an equation, and that he had effected measurements of the circle and regular polygons. Of these the circle measurement deserves a further mention.

The problem of circle measurement begins by no means in Japan with Seki. The problem had been studied from before his time. It

was he who first considered the matter in an analytical manner, as it is usually stated.

Seki died in 1708 and four years later some of his writings were published with the title of the Kwatsuyō Sampō, in which one of his results was given. Here no analytical process is followed. A certain process of rectification of the value obtained is tried.

The Kwatsuyō Sampō goes no further than this. But the same way of rectification employed in it may be applied any number of times, when the final result becomes more and more minute; which was realised by Takebe in his Fukyū Tetsujutsu of 1722. Takebe also derives various fractional values of π, applying to successive divisions.

In the above however we see nothing analytical. But such appeared soon.

Matsunaga's Hōyen Sankyō of 1739 is a manuscript work that was highly valued in old times. In this work there is given a series for the square of an arc, the necessary calculation not being described. Such a calculation is found in detail in the Yenri Tetsujutsu written by Takebe, who tries to carry his manipulation by bisecting the arc and taking recourse to a kind of incomplete induction. The invention of the method followed by Takebe is generally ascribed to Seki, which is however doubtful. According to certain authorities, Takebe received a secret writing of Seki from his master's son, when the latter, being dismissed from service, was living under his care, and it was on this that he had based the considerations in his manuscript. In that case Takebe could not have possessed of the process, he describes, previous to 1735. A nonsense, as Takebe had already known of the series before us as early as 1722, when he wrote the Fukyū Tetsujutsu. It is even mentioned in the Yenri Hakki of 1729 that the process had been obtained by Takebe in consequence of perseverance through tens of years.

In the Kenkon no Maki, whose author is not known, Takebe's process had become exceedingly simplified.

Some twelve years since the publication of the Shūki Sampō, in which the tenzan method was made public for the first time, there appeared the Seiyō Sampō (1779) of Fugita (1734—1807), a work, where indeed the tenzan way of solutions are not given, but which has since served for a standard text book of the tenzan algebra. It did very much in popularising the science. Fugita wrote a great many of works, that were read in manuscripts. He had been a very renowned

man highly respected, when he lived. His mean, underhand spirit had caused him, however, to enter into a quarrel with Aida (1847—1817), who was a man naturally proud and destitute in modesty, but who was a man of genius. We cannot dwell upon the particulars of the controversy that ensued. Suffice to say, it had been the collision of academic feud and unprotected genius.

Aida, although so loose in character as he was, made numerous discoveries, of which we may mention, by way of an example, his general solution of the indeterminate equation, $x^2 + y^2 + z^2 + u^2 = \varrho^2$.

The indeterminate equations were much studied by Aida and other scholars. Gokai's solution of the equation $x^3 + y^3 + z^3 = u^3$ was given in the Shamei Sampu of 1827 published by Shiraishi.

Aida carried an extensive study on the ellipse, a subject that had since attracted the attention of scholars in Japan. It is worthy of notice that the ellipse has been studied in Japan exclusively as the section of a circular cylinder, and never as that of a cone. It comes therefore that neither parabola nor hyperbola was considered by the Japanese of the old school. About the ellipse too we are not left with any writings that concern to its foci. Aida employs a mode of projection in his studies on the ellipse.

Ajima (1739—1798) was a contemporary of Aida and Fujita. He loved seclusion and tranquility; he lived in quiet ever pursuing after truths. Among Ajima's numberless researches that which concerns to a problem suspended in a temple at Kyōto is one of the most renowned. It relates to the magnitudes that arise in connection with a circle and a square inscribed in a circular segment across its sagitta. The original solution of it, as obtained by Tsuda, had been of the 1024$^{\text{th.}}$ degree, while it was reduced by Nakata to an equation of the degree 46. Ajima's result consisted of one of only 10$^{\text{th.}}$ degree. That solution was obtained in 1773.

Ajima solved also the problem of a triangle inscribed with three circles, which is nothing but Malfatti's problem. There are various problems attacked by Ajima, in which numerous circles are inscribed in a circle. Problems of these kinds, that relate to geometrical figures, were extensively studied by Japanese mathematicians of old. But these were treated only in algebraical ways; they were never studied from geometrical point of view. Geometry as such never presented itself to the Japanese mind. The science as a demonstrative system has first become intelligible to the Japanese only after they have got quite recently in direct contact with the Occidental civilization.

In addition to what has been said above, Ajima had secured a step, and a firm step, in the advance of the circular principle. Ajima did not divide an arc into equal parts as his predecessors had done. On the contrary he cuts the chord of the arc into a number of equal parts. Normal chords being drawn at the points of division, their lengths are easy of calculation. Proceeding from such he derived the expression for the area bounded between two equal, oppositely situated arcs.

This success brought Ajima to carry his step on the evaluation of the volume of a solid got by piercing a circular cylinder by another orthogonally through its middle. This was done first expressing the area of a section normal to the axis of the larger cylinder, and then applying the process of integration, so to say, for a second time. In the process a double series was employed indeed, but it involved no double integral, for the two integrations were separately carried. Ajima's manuscript on the subject bears the date of 1794.

Ajima's success in this one problem did not fail to arouse numerous scholars to direct their attentions to problems of the same or like nature. Japanese mathematics soon became full and rich with studies in that domain.

Ajima's pupil Sakabe (1759—1824) and Sakabe's pupil Kawai, made together various investigations. Sakabe invented in 1803 a process of solving the cubic equation on the soroban, that is, by repeated applications of the extraction of a square root, the resulting formula leading to an expression resembling the continued fraction. The Kaishiki Shimpō was published in the same year by Kawai, although it is believed to be a writing by his master. Here there are given a process by which all real roots, positive and negative, of a numerical equation can be found approaching step by step.

Sakabe and Kawai worked on the rectification of the ellipse, and obtained various formulae during 1813—1822. But these are all complicated. It was also the same with Ichino's result, a pupil of Aida, who carried his study on the basis of a manuscript of Kawai.

Sakabe published in 1810—1815 the Tenzan Shinan-roku, a systematic treatise on the tenzan method. The work had since become very popular among mathematicians. The Sampō Shinsho, written by Hasegawa, was published in 1830 by his pupil's name. It went through several editions.

The rectification of the ellipse was very much simplified by Wada. In the Calculation of the circle he took recourse to the differential of

an arc, which completed the sound establishment of the process began by Ajima. Wada's method is recorded in the Yenri Shinkō of 1818. Wada made advantage of this method to apply it to the case of an ellipse; he obtained his result in a simple form. He calculated a table, by means of which integration or *folding* as the Japanese termed could be effected. Wada's study extended to a wide scope of problems. He considered cycloid and other curves.

Shiraishi's Shamei Sampu, that appeared in 1827, was the first instance of the complicated problems in the circular principle being published. There is given among others a formula for the surface of an ellipsoid. The solutions of problems of this kind were published by Iwai in his Yenri Hyōshaku of 1837. Hasegawa's Gyoku-seki Tsūkō appeared in 1844 by his pupil's name, a well organised treatise on the circle-principle.

We shall not meddle any further with the particulars of the studies of the old Japanese. But there is a subject we cannot forget to describe. It consists of the ruikan-jutsu, perhaps due to Wada. Problems solved by that process were published in Saitō's Yenri-kan of 1834. One of the problems is to find the diameter of a circle, when, its arc whose sagitta is given should have a minimum value. The problems of this kind were attacked by Japanese scholars by a process in which the real value is approached step by step.

The catenary was also studied in Japan, and various scholars obtained their own results, while the solution of Ōmura was by far the most remarkable of all. He solved it by applying to the ruikan-jutsu.

The centre of gravity caught a deep attention in Japan, and especially in later years. The last and the most complicated of problems on this subject was given by Hagiwara in his Yenri San-yō of 1878.

T. ENDO, ON THE EXTRACTION OF CUBE ROOT.[1]

In the following we give three ways of extracting the cube root of a given number.

We have the identical relations:

$$1^3 = 1,$$
$$2^3 = 8 = 1 + (1 + 6),$$
$$3^3 = 27 = 1 + (1 + 6) + (1 + 6 + 12),$$
$$4^3 = 64 = 1 + (1 + 6) + (1 + 6 + 12) + (1 + 6 + 12 + 18),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

1) The Journal of the Society of Mathematics, Tokyo. Vol. 4, pp. 435—442, 1890.

Hence if we subtract from a given number N 1, $1 + 1 \cdot 6$, $1 + 1 \cdot 6 + 2 \cdot 6$, $1 + 1 \cdot 6 + 2 \cdot 6 + 3 \cdot 6$, $\ldots$, $1 + 1 \cdot 6 + 2 \cdot 6 + \ldots + m \cdot 6$ in succession, and if we find the last remainder thus obtained less than $1 + \{1 + 2 + 3 + \ldots + (m+1)\}\, 6$, then $m + 1$ will be a first approximation to the cube root of N.

This is the first of our three ways.

The numbers in the above table admit themselves to be rewritten as follows:

$$\begin{aligned} 1^3 &= 1, \\ 2^3 &= 2 + 1 \cdot 6, \\ 3^3 &= 3 + (1 + 2)\, 6, \\ 4^3 &= 4 + (1 + 2 + 3)\, 6, \\ &\ldots\ldots\ldots \end{aligned}$$

From this it will appear that the number N, divided by 6, will give a rest which, added to a multiple of 6, will afford a value to the cube root of N. For such a multiple we shall have $\{m\, 6\}^3 < N < \{(m+1)\, 6\}^3$, by virtue of which the appropriate value of m could be calculated without intricating any difficulty.

This way will be seen somewhat simpler than the one first given.

Our third way is this:

Since we have the identity

$$n^3 - m^3 = \{1 + (1 + 2 + \ldots \overline{m+1})\, 6\} + \ldots + \{1 + (1 + 2 + \ldots + n)\, 6\},$$

if the number m is found in some way or other, the next steps that follow might be taken quite the same as in the first case.

For example, we take $N = 804{,}357$. The calculation will be carried as in the scheme:

$$\begin{array}{rrr} & & 804357 \\ (90^3 =) & & \underline{729000} \\ & & 75357 \\ 91 \times 90 \times 3 + 1 = 24571 & \ldots & \underline{24571} \\ 91 \times 6 \;\;.\;\;.\;\;\; \underline{546} & & 50786 \\ 25117 & \ldots & \underline{25117} \\ 92 \times 6 \;\;.\;\;.\;\;\; \underline{255} & & 25669 \\ 25669 & \ldots & \underline{25669} \\ & & 0 \end{array}$$

$$\therefore \sqrt[3]{804357} = 92 + 1 = 93.$$

ON A QUEER NUMBER.

I.

The Journal of the Physics School in Tokyo, Vol. 5, p. 82 (February, 1896), contains a note under the heading of *A Queer Number.* The note runs thus:

When, in any system of notation, a number is formed by arranging all the figures in their natural order, and the number thus obtained is multiplied by a number less by 2 than the radix of the system and is added by a number 1 less than the radix, then the resulting number will be one whose figures proceed in the reverse order.

In the scale of 7, for example, we have

$$123456 \times 5 + 6 = 654321;$$

in that of 10,

$$123456789 \times 8 + 9 = 987654321;$$

and if we take 13 for the basis of notation,

$$123 \ldots 9\,(X)(XI)(XII) \times (XI) + (XII) = (XII)(XI)(X)\,98 \ldots 321.$$

In any other notation too the same property will always reveal itself.

In the next number of the same *Journal*, pp. 99—103, three different ways of demonstration for the above interesting property of a number were published.

We reproduce each of them in the following lines:

1. An anonymous proof under the sign of K. K.

If we denote the radix of a scale of notation by r, the number, that is formed by all the figures arranged in their natural order, will have the expression

$$\varphi(r) = r^{r-2} + 2r^{r-3} + 3r^{r-4} + \ldots + (r-3)r^2 + (r-2)r + (r-1)$$

or

$$= (r-1) + (r-2)r + (r-3)r^2 + \ldots + 3r^{r-4} + 2r^{r-3} + r^{r-2}.$$

The number, which we obtain by reversing the order of figures, is this

$$\psi(r) = (r-1)r^{r-2} + (r-2)r^{r-3} + \ldots + 2r + 1$$
$$= 1 + 2r + 3r^2 + \ldots + (r-1)r^{r-2}.$$

It is only sufficient for us to prove that the equality

$$(r-2)\varphi(r) + (r-1) = \psi(r)$$

holds correct for any integral value of r.

In the formula

$$S = a + (a+b)\,r + (a+2b)\,r^2 + \ldots + (a + \overline{a-1}\,b)\,r^{n-1}$$

$$= \frac{1}{r-1}\left[(a+\overline{n-1}\,b)\,r^n - a - \frac{b\,r\,(r^{n-1}-1)}{r-1}\right],$$

we make the substitutions $a = r-1$, $b = -1$, $n = r-1$, and we get

$$\varphi(r) = \frac{1}{r-1}\left\{r^{r-1} - (r-1) + \frac{r^{r-1}-r}{r-1}\right\};$$

while the substitutions $a = 1$, $b = 1$, $n = r-1$ in the same formula give us

$$\psi(r) = \frac{1}{r-1}\left\{(r-1)\,r^{r-1} - 1 - \frac{r\,(r^{r-2}-1)}{r-1}\right\}.$$

These verify immediately the formula

$$(r-2)\,\varphi(r) + (r-1) = \psi(r).$$

2. R. Ichikawa's proof.

We take

$$S = m^{m-2} + 2m^{m-3} + 3m^{m-4} + \ldots + (m-2)\,m + (m-1)$$

for a number under question, m denoting the radix of scale.

Subtracting S from m times of itself we get

$$S\,(m-1) = m^{m-1} + m^{m-2} + \ldots + m^2 + m - (m-1).$$

S being subtracted from this for a second time, it remains

$$S\,(m-2) = (m-1)\,m^{m-2} + (m-2)\,m^{m-3} + \ldots + 2m + 1 - (m-1).$$

It follows therefore the relation

$$S\,(m-2) + (m-1) = (m-1)\,m^{m-2} + (m-2)\,m^{m-3} + \ldots + 2m + 1,$$

which expresses the property before us.

3. Proof by S. Tamano.

In the identity

$$\frac{(a-1)\,(a-b)}{a} = \{a - (1+b)\} + \frac{b}{a},$$

where a and b represent whole numbers, of which a is the greater of the two, we put successively $b = 1, 2, 3, \ldots, (a-1)$ and we get

$$\begin{aligned}
(a-1)(a-1) &= a\,(a-2) + 1,\\
(a-1)(a-2) &= a\,(a-3) + 2,\\
(a-1)(a-3) &= a\,(a-4) + 3,\\
&\ldots\ldots\ldots\ldots\\
(a-1)(a-\overline{a-1}) &= a\,(a-a) + (a-1);
\end{aligned}$$

or what is the same thing,

$$\begin{aligned}
&(a-2)(a-1)+(a-1) &&= a(a-2)+1,\\
&(a-2)(a-2)+(a-2) &&= a(a-3)+2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &&\\
&(a-2)(a-\overline{a-1})+(a-\overline{a-1}) &&= a(a-a)+(a-1).
\end{aligned}$$

Multiply these equalities respectively by 1, a, a^2, . . ., a^{a-2}, and add them together. We then obtain

$$(a-2)\{1a^{a-2}+2a^{a-3}+3a^{a-4}+\ldots+(a-2)a+(a-1)\}+(a-1)$$
$$=(a-1)a^{a-2}+(a-2)a^{a-3}+\ldots+2a+1,$$

the very formula required to be established.

II.

In conjunction to the appearance of the studies we have reproduced in Part I, T. Hayashi published two papers on the same or associated subjects shortly afterwards. These papers are entitled *On some properties of numbers that arise by arranging all the figures in their natural and reverse orders in any scale of notation* and *A property of the number formed by arranging all the figures in their natural order,* both of which appeared in *the Journal of the Tokyo Physics School, Vol. 5, pp. 153—156 and 266—267, May and September, 1896.*

The contents of these papers are essentially as follows:

Theorem 1. In the scale of r, we have

$$1\,2\,3\ldots[r-1]\times(r-1)+r=1\,1\,1\ldots1 \text{ (}r\text{ figures)}.$$

For

$$r^{r-2}+2r^{r-3}+\ldots+(r-2)r+(r-1)=\frac{1}{r-1}\left\{r^{r-1}-(r-1)+\frac{r^{r-1}-r}{r-1}\right\},$$

$$\therefore \{r^{r-2}+2r^{r-3}+\ldots+(r-1)\}(r-1)+r=r^{r-1}+1+\frac{r^{r-1}-r}{r-1}$$
$$=\frac{r^r-1}{r-1}=r^{r-1}+r^{r-2}+\ldots+r+1.$$

Theorem 2. In the scale of r

$$1\,2\,3\ldots[r-1]+[r-1][r-2]\ldots3\,2\,1=1\,1\,1\ldots1 \text{ (}r\text{ figures)}.$$

For the sum of two complementary numbers[1]) is equal to the radix of the scale.

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1) A number is said to be complementary to another when the former is obtained by subtracting the ratter from the radix.

Theorem 3. In the scale of r

$$1\,2\,3\ldots[r-2][r-1]\times(r-1)+(r-1)=[r-1][r-2]\ldots3\,2\,1.^{1)}$$

Proof obvious from theorems 1 and 2.

Theorem 4. In the scale of 2s the sum of all the figures is equal to $[s-1][s]$, *while in the scale of* $2t+1$ *the same sum is equal to* $[t]0$.

For

$$\begin{aligned}1+2+3+\ldots+(2s-2)+(2s-1)&=s(2s-1)\\&=(s-1)2s+s\\&=[s-1][s]\end{aligned}$$

and

$$1+2+3+\ldots+(2t-1)+2t=t(2t+1)=[t]0.$$

Theorem 5. In the scale of 2s

$$\begin{aligned}1\,2\,3\ldots[2s-1]\times\{1+2+3+\ldots+(2s-1)\}&\\+\{1+2+3+\ldots+(2s-1)+2s\}&\\=s\,s\,s\ldots s\;(2s\text{ figures})+s.&\end{aligned}$$

For

$$\begin{aligned}&1\,2\,3\ldots[2s-2][2s-1]\times\{1+2+3+\ldots+(2s-1)\}\\&\qquad+1+2+3+\ldots+(2s-1)+2s\\&=s\{1\,2\,3\ldots[2s-2][2s-1]\times(2s-1)+2s\}+s\\&=s\times1\,1\,1\ldots1+s\qquad\text{by theor. 1,}\\&=s\,s\,s\ldots s+s.\end{aligned}$$

Theorem 6. In the scale of r

$$\begin{aligned}&[r-1][r-2]\ldots4\,3\,2\,1\times(r-1)-1\\&\qquad=[r-2][r-2]\ldots[r-2]\quad(r\text{ figures}).\end{aligned}$$

Easy to prove by theorems 1 and 2.

Theorem 7. In the scale of 2s

$$\begin{aligned}&[2s-1][2s-2]\ldots3\,2\,1\times\{1+2+3+\ldots(2s-1)\}-1\\&\qquad=[s-1][s-1]\ldots[s-1]\quad(2s+1\text{ figures}).\end{aligned}$$

For by theorems 2 and 5 the left-hand member will be seen to be equal to the number constituted by arranging s $2s$ times and multiplying by $2s-2$, the result being added by $s-1$. Hence if we arrange s $2s$ times, multiply it by 2 and add 1, the result will be $1\,1\,1\ldots1$ $(2s+1$ figures).

Theorem 8. In the scale of 2s

$$[2s-1][2s-2]\ldots3\,2\,1\times s=[s-1]s[2s-1]\ldots2[s+2]1[s+1]0s.$$

1) This is the proposition considered in Part I.

For

$$[2n+1] \times s = [n][s]$$

and

$$[n] \times s = [n]0.$$

Theorem 9. In the scale of $2s$ *the numbers* $1\,2\,3 \ldots [2s-1]$ *and* $[2s-1][2s-2] \ldots 3\,2\,1$ *are perfectly divisible by* $2s-1$.

For if the number arranged in the natural order of figures is divisible by a number, the number in the reverse order will also be so by virtue of the theorems 1 and 2. But

$$1+2+3+\ldots+(r-2)+(r-1)=\frac{r}{2}(r-1)$$

being subtracted from

$$r^{r-2}+2r^{r-3}+3r^{r-4}+\ldots+(r-2)r+(r-1),$$

we have

$$(r^{r-2}-1)+2(r^{r-3}-1)+3(r^{r-4}-1)+\ldots+(r-2)(r-1)$$

remaining, that is obviously divisible by $r-1$.

For an even value of r the subtrahend too will be divisible by $r-1$, and our theorem at once follows.

Theorem 10. If the number of scale be written in the form $mn+1$, *and* m *(or* n*) be multiplied to the number formed by arranging all the figures in their natural order, and* m *(or* n*) be added to the product, the resulting number has a period that consists of* n *(or* m*) figures.*

In the scale of 11, for example, we have

$$11 = 2 \times 5 + 1,$$

$$1\,2\,3 \ldots 9[10] \times 2 + 2 = 24690,\ 24690,$$

$$1\,2\,3 \ldots 9[10] \times 5 + 5 = 60,\ 60,\ 60,\ 60,\ 60.$$

The proof easily follows from theorem 1. It will be carried specially for the case of the scale of 11.

In that case we have

$$1\,2\,3 \ldots 9[10] \times 10 + 10 = 11 \ldots 10 \quad \text{(ten 1's)},$$

i. e.

$$1\,2\,3 \ldots 9[10] \times 2 \times 5 + 2 \times 5 = 11 \ldots 10.$$

$$\therefore 1\,2\,3 \ldots 9[10] \times 2 + 2 = 111 \ldots 10 \div 5,$$

$$1\,2\,3 \ldots 9[10] \times 5 + 5 = 111 \ldots 10 \div 2.$$

But 10, 110, 1110, 11110 cannot be divided by 5 and 111110 is exactly divisible by 5; and 10 is not divisible by 2, while 110 can be divided by 2. The right-hand sides of the above equalities therefore consist of numbers that have the periods of 2 and 5 figures respectively.

The same way of demonstration equally applies for the general case.

ANOTHER QUEER NUMBER.

1. In a note in the *Journal of the Physics School in Tokyo* it was once given that the number

$$526315,789473,684210,$$

multiplied with 3, becomes

$$1,578947,368421,052630,$$

where the figures in the original number all appear only cyclically interchanged. It is also stated that in place of multiplication with 3, the same number may be multiplied with 4 or 8, or divided by 5, when like results are expected.

This queer property of the number instigated some studies on the subject, when two papers were made public, one by T. Hayashi[1]) and the other by U. Fujimaki.[2])

In this place we give the results of these studies.

2. *Hayashi's explanation and generalization* of the property above mentioned.

About the number A, whose form is

$$A = 10 + r(10)^2 + r^2(10)^3 + r^3(10)^4 + \dots$$

or

$$= 10\{1 + 10r + (10r)^2 + (10r)^3 + \dots\},$$

the following theorems may be derived:

Theorem 1. The number A is periodic, and the figures contained in one of its periods is $9r-2$ or $9r-1$ according as r is even or odd.

Theorem 2. When A is multiplied with or divided by any number, the result is always periodic, the number of figures in one period not changing and all the same figures appearing in an order cyclically interchanged.

Theorem 3. When one period of A is multiplied with or divided by any number, the figures will be only interchanged cyclically.

Theorem 4. As the number of figures in one period of A is always even, if we divide them into two parts and add them together, the result will be a number formed by arranging the one and same figure 9.

The proof of these propositions is not difficult, so that it is left to the reader.

The properties here described are evidently applicable to the case with any scale of notation

1) *On a number that changes its figures only cyclically when multiplied or divided by any number.* Vol. 6, pp. 148—149, May, 1897.

2) *On some queer numbers.* Vol. 7, pp. 16—21, December, 1897.

3. *Fujimaki's result.*

When m and n are integers and $\frac{10^m-1}{n}$ is divisible without remainder, the integral quotient being a number of m digits, then numbers that will be obtained by cyclically interchanging the order of its figures are all multiples of the original number.

If the figures in the quotient of $10^m - 1 = 999\ldots9$ divided by n will be denoted by $a_1, a_2, \ldots, a_m$, we have

$$\frac{10^m-1}{n} = a_1 a_2 \ldots a_m, \tag{1}$$

and it is to be proved that the number

$$a_{r+1}\, a_{r+2} \ldots a_m a_1 a_2 \ldots a_r$$

is a multiple of $a_1 a_2 \ldots a_m$ for any value of r from 1 to $m-1$.

First let be assumed

$$\begin{aligned} \frac{10^m-1}{n} \times x &= a_{r+1}\, a_{r+2} \ldots a_m a_1 a_2 \ldots a_r \\ &= a_{r+1}\, a_{r+2} \ldots a_m \times 10^r + a_1 a_2 \ldots a_r. \end{aligned} \tag{2}$$

Then from (1)

$$\frac{10^m-1}{n} = a_1 a_2 \ldots a_r \times 10^{m-r} + a_{r+1}\, a_{r+2} \ldots a_m,$$

$$\therefore\ \frac{10^m-1}{n} \times 10^r = a_1 a_2 \ldots a_r \times 10^m + a_{r+1}\, a_{r+2} \ldots a_m \times 10^r. \tag{3}$$

(2) being subtracted from (3),

$$\frac{10^m-1}{n}(10^r - x) = a_1 a_2 \ldots a_r \times (10^m - 1),$$

or

$$10^r - x = a_1 a_2 \ldots a_r \times n.$$

But $a_1 a_2 \ldots a_r$ is obviously the quotient of the first r figures in $10^m - 1 = 99\ldots9$; the remainder in that case being denoted by R_r, we have

$$10^r - 1 = a_1 a_2 \ldots a_r \times n + R_r,$$

$$\therefore\ a_1 a_2 \ldots a_r \times n = 10^r - (R_r + 1),$$

or

$$10^r - x = 10^r - (R_r + 1),$$

$$\therefore\ x = R_r + 1.$$

Thus $a_{r+1}\, a_{r+2} \ldots a_m a_1 a_2 \ldots a_r$ is proved to be a multiple of $\frac{10^m-1}{n} = a_1 a_2 \ldots a_m$; and the coefficient of this multiple is one more than the remainder obtained after the r^{th} operation, when $10^m - 1 = 99\ldots9$ is divided by n.

Note 1. When n is a number of one digit, $\frac{10^m - 1}{n}$ is of m digits, and the above reasoning directly applies. When n is a number of p digits, the same quotient is of $m - p + 1$ digits. In this case the above property will apply if we form a number by adding $p - 1$ zeros to the right of the figures in the quotient.

Note 2. When n is a prime number, $\frac{10^{n-1} - 1}{n}$ is an integer by Fermat's theorem, so that we know that there are an infinity of numbers that possess the above property.

T. HAYASHI, ON THE EXAMINATION OF PERFECT SQUARES AMONG NUMBERS FORMED BY THE ARRANGEMENTS OF THE NINE EFFECTIVE FIGURES.[1]

There are 9! different ways of forming numbers of 9 digits with the 9 figures. How many squares will arise among these? This question was set forth some years ago by Artemas Martin in a British journal, and Biddle gave his answer as to have 29 such numbers.[2]

Problems of this kind does not seem, save some few exceptions, to be soluble without being applied to the tediousness of practical calculations.

It will be treated here how we can abridge such a numerical treatment.

Numbers of nine digits, that are formed of the nine effective figures, can be evidently divided by 9, and the least and greatest of them are 123456789 and 987654321, which are 9 times of 13717421 and 109739369 respectively, so that the square roots of $\frac{1}{9} \times$ (th eperfect squares) lie between

$$\sqrt{13717421} = 3704 \quad \text{and} \quad \sqrt{109739369} = 10475.$$

If the square of a number, that lies between these two numbers, is multiplied by 9 and gives a number composed of 9 different figures, zero not inclusive, then this number is evidently one of those that are required.

To multiply a number by 9, we have to arrange the number and to subtract itself lower by one digit, and the squares of numbers of

1) Journ. of Phys. Sch., Vol. 5, pp. 203—206, July, 1896.

2) Given in the same Journal, Vol. 5, p. 171.

four digits will be found in Barlow's table or in that of Hutton. We can therefore make out the required perfect squares by examining the nearly 7000 numbers.

But a little more abridgment.

First we shall examine the last two figures of the squares of numbers from 3704 to 10475. Every square number should end in one of the following 22 sets:

(00); (01), 21, 41, 61, 81;

04, 24, 44, 64, 84; 25;

09, 29, 49, 69, (89); (16), 36, (56), 76, (96);

because they arise from

$$(10m+n)^2 = 100m^2 + 20mn + n^2.$$

Of these sets those that are included within brackets are to be rejected, because they raise a zero or cause a double figure when multiplied with 9.

The last two figures of a square number underlie the following rule. The arrangement of these for the squares of numbers with 01 to 24 for their last figures are in the reverse order, in the same or in the reverse again respectively, of the sets for those of the squares of numbers whose last two figures are from 26 to 49, 51 to 74 or 76 to 99. For we have

$$\{m(10)^2 + (50+n)\}^2 - \{m(10)^2 + n\}^2 = 100A \qquad (n < 50),$$

$$\{m(10)^2 + (25+n)\}^2 - \{m(10)^2 + (25-n)\}^2 = 100B \quad (n < 25),$$

$$\{m(10)^2 + (75+n)\}^2 - \{m(10)^2 + (75-n)\}^2 = 100C \quad (n < 25).$$

Thus the last two figures of all numbers are to be examined only in conjunction with 25 numbers that are to be squared; and we reject those that end with the sets enclosed in brackets in the annexed table.

To be squared	squares	To be squared	squares	To be squared	squares
(01)	(01)	09	81	(17)	(89)
(02)	(04)	(10)	(00)	18	24
03	09	(11)	(21)	19	61
(04)	(16)	12	44	(20)	(00)
05	25	13	69	21	41
06	36	14	96	22	84
07	49	15	25	23	29
08	64	(16)	(56)	24	76

Next we can try a similar process with the three last figures, whereby we find a period with every 250 numbers.

Further abridgment goes proceeding in the same manner, but the work remains for the trial of a practical calculator.

T. HAYASHI, A CHINESE THEOREM ON PRIME NUMBERS.[1])

The proposition: *The quantity $2^n - 2$ is or is not divisible by n according as n is a prime number or not,* is said to originate with the Chinese, a proposition that is recorded perhaps for the first time in the Western World by W. W. R. Ball in his *Mathematical Recreations and Problems;* while it has been handed down in China, as is stated, as early as from the time even of Chou Kong. We are however all at a loss in what book or books to look for it, both Chinese as well as in any other Oriental language.

The first half of the said proposition is no other than Fermat's theorem, so that no room for any doubt about its authenticity. As to the latter part Ball declares he does not yet procure any proof.

As we perceive, this latter part is not correct. For, n being taken for the product of two odd prime numbers p and q, the proposition may be proved not to hold for some special cases.

1. If two numbers p and q can be found such that

$$2^p \equiv 2, \quad 2^q \equiv 2 \pmod{pq},$$

then the quantity $2^{pq} - 2$ will be divisible by pq.

But

$$2^{10} = 1 + 3 \cdot 31 \cdot 11,$$

$$\therefore\ 3^{30} \equiv 1 \pmod{31 \cdot 11},$$

that is,

$$2^{31} \equiv 2;$$

and

$$2^{11} = 2 + 6 \cdot 31 \cdot 11,$$

or

$$2^{11} \equiv 2 \pmod{31 \cdot 11}.$$

Hence for $p = 31$ and $q = 11$, that is, for $n = 341$, the quantity $2^n - 2$ is divisible by n.

1) Journ. of Phys. Sch., Vol. 9, pp. 143—144, March, 1900.

2. $2^{pq}-2$ will be divisible by pq, if

$$2^{p-1}\equiv 1,\quad 2^{q-1}\equiv 1 \pmod{pq},$$

and, therefore, if $p-1$ and $q-1$ have a common factor, p' say, for which

$$2^{p'}\equiv 1 \pmod{pq}.$$

But

$$2^{11}=1+23\cdot 89,$$

or

$$2^{11}\equiv 1 \pmod{23\cdot 89},$$

and 11 is a factor common to $23-1$ and $89-1$. Hence for $p=23$ and $q=89$, i. e., for $n=2047$, 2^n-2 is divisible by n.

Thus the latter part of our proposition cannot be true in general, because it gives cases where it does not hold.

T. HAYASHI, ON THE RESIDUES OF NUMBERS THAT CONSTITUTE PASCAL'S TRIANGLE WITH RESPECT TO A PRIME NUMBER.[1])

The same reasoning, that Kurt Hensel serves himself in his paper on *the extension of Fermat's theorem and Wilson's theorem* published in the *Archiv der Mathematik und Physik* of July, 1901, may be equally applied for the deduction of some theorems that concern to the residues of numbers that form the triangle of Pascal in respect to a prime number.

Let p be a prime number and q a number that lies between 1 and $p-1$, inclusive. Here evidently it will be

$$\binom{p}{q}=\frac{p(p-1)\,\ldots(p-q+1)}{q!}\equiv 0 \pmod{p}.$$

$$\therefore\ (x+a)^p\equiv x^p+a^p \pmod{p}.$$

Now put $p=\mu+\nu$, μ and ν being any positive integers, and we shall have

$$(x+a)^\mu\,(x+a)^\nu\equiv x^p+a^p,$$

that is,

$$x^\mu\,(x+a)^\nu\equiv(x^p+a^p)\cdot x^\mu\,(x+a)^{-\mu},$$

or, both sides being expanded,

1) Journal of Physics School, Vol. 10, pp. 391—392, October, 1901.

$$
\begin{aligned}
x^p + \binom{\nu}{1} a x^{p-1} + \binom{\nu}{2} a^2 x^{p-2} + \dots \\
\equiv x^p - \begin{bmatrix}\mu\\1\end{bmatrix} a x^{p-1} + \begin{bmatrix}\mu\\2\end{bmatrix} a^2 x^{p-2} + \dots \\
+ \left\{ - \begin{bmatrix}\mu\\p\end{bmatrix} + 1 \right\} a^p + \left\{ + \begin{bmatrix}\mu\\p+1\end{bmatrix} - \begin{bmatrix}\mu\\1\end{bmatrix} \right\} a^{p+1} x^{-1} \\
+ \left\{ - \begin{bmatrix}\mu\\p+2\end{bmatrix} + \begin{bmatrix}\mu\\2\end{bmatrix} \right\} a^{p+2} x^{-2} + \dots,
\end{aligned}
$$

where

$$\binom{r}{s} = \frac{r(r-1)(r-2)\dots(r-s+1)}{s!},$$

$$\begin{bmatrix}r\\s\end{bmatrix} = \frac{r(r+1)(r+2)\dots(r+s-1)}{s!}.$$

A comparison of the coefficients of like powers of x will show these relations

$$
\left.\begin{aligned}
\begin{bmatrix}\mu\\1\end{bmatrix} &\equiv -\binom{\nu}{1} \\
\begin{bmatrix}\mu\\2\end{bmatrix} &\equiv +\binom{\nu}{2} \\
\dots&\dots \\
\begin{bmatrix}\mu\\\nu-1\end{bmatrix} &\equiv (-1)^{\nu-1}\nu \\
\begin{bmatrix}\mu\\\nu\end{bmatrix} &\equiv (-1)^{\nu}
\end{aligned}\right\},\quad
\left.\begin{aligned}
\begin{bmatrix}\mu\\\nu+1\end{bmatrix} &\equiv 0 \\
\begin{bmatrix}\mu\\\nu+2\end{bmatrix} &\equiv 0 \\
\dots&\dots \\
\\
\begin{bmatrix}\mu\\p\end{bmatrix} &\equiv 0
\end{aligned}\right\},\quad
\left.\begin{aligned}
\begin{bmatrix}\mu\\p+1\end{bmatrix} &\equiv -\binom{\nu}{1} \\
\begin{bmatrix}\mu\\p+2\end{bmatrix} &\equiv +\binom{\nu}{2} \\
\dots&\dots \\
\begin{bmatrix}\mu\\p+\nu-1\end{bmatrix} &\equiv (-1)^{\nu-1}\nu \\
\begin{bmatrix}\mu\\p+\nu\end{bmatrix} &\equiv (-1)^{\nu}
\end{aligned}\right\}.
$$

Or to enunciate in a general way, r being any integer, it runs:

I. $$\begin{bmatrix}\mu\\rp+s\end{bmatrix} \equiv (-1)^s \binom{\nu}{s} \pmod{p},$$

where s denotes a whole number from 1 to ν;

II. $$\begin{bmatrix}\mu\\rp+s\end{bmatrix} \equiv 0 \pmod{p},$$

where s lies between $\nu+1$ and $p-1$, inclusive;

III. $$\begin{bmatrix}\mu\\rp\end{bmatrix} \equiv 1 \pmod{p}.$$

By an application of the identical relation

$$\begin{bmatrix}\mu\\t\end{bmatrix} = \binom{\mu+t-1}{t},$$

these congruences let themselves rewrite as follows:

$$\text{I}'. \quad \binom{rp+\mu+s-1}{rp+s} \equiv (-1)^s \binom{\nu}{s} \quad (s=1,\,2,\,3,\,\ldots\,\nu),$$

$$\text{II}'. \quad \binom{rp+\mu+s-1}{rp+s} \equiv 0 \quad (s=\nu+1,\ \nu+2,\,.\,.,\ p-1),$$

$$\text{III}'. \quad \binom{rp+\mu-1}{rp} \equiv 1.$$

These congruences may be looked upon as generalizations of the theorem given in *Lucas' Théorie des Nombres, pp. 419—420.*

By putting $p=\mu-\nu$ we could else come across a number of new congruences.

T. KARIYA, ON THE SUMS OF POWERS OF NATURAL NUMBERS.[1]

I.

1. If S_m denote the sum of the m^{th} powers of $1,\,2,\,3,\,\ldots,\,n$, there exist the formulae,

$$2S_5+S_3=3S_2{}^3, \quad S_7+S_5=2S_3{}^2,$$

that are due to Jacobi, as we see from the *Briefwechsel zwischen Gauss und Schumacher.*

The following treatment leads us to like results.

In the figure

1^m	2^m	3^m	$\cdots$	n^m
2^m	4^m	6^m	$\cdots$	$(2n)^m$
3^m	6^m	9^m	$\cdots$	$(3n)^m$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
n^m	$(2n)^m$	$(3n)^m$	$\cdots$	$(nn)^m$

the sum of the r^{th} line is

1) The Journal of the Tokyo Phys. Sch., Vol. 16, pp. 201—203, 241—244, 1907. Results of two papers.

$$r^m + (2r)^m + (3r)^m + \ldots + (nr)^m = r^m S_m,$$

so that the whole sum will be represented by

$$S_m(1^m + 2^m + 3^m + \ldots + n^m) = S_m^2.$$

If we have to sum along the lines indicated in the figure, we shall have for the p^{th} of such lines

$$p^m + (2p)^m + \ldots + \{(p-1)p\}^m + (pp)^m + \ldots + p^m.$$

Consequently

$$\text{the whole sum} = \sum_{p=1}^{n}\left\{2p^m \sum_{r=1}^{p} r^m - p^{2m}\right\}.$$

$$\therefore \sum_{p=1}^{n}\left\{2p^m \sum_{r=1}^{p} r^m - p^{2m}\right\} = (S_m)^2,$$

which is an extended form of Jacobi's formulae.

In particular for $m = 1, 2, 3, \ldots$, we have

(1) $S_3 = S_1^2$, (2) $2S_5 + S_3 = 3S_2^2$,

(3) $S_7 + S_5 = 2S_3^2$, (4) $6S_9 + 10S_7 - S_5 = 15S_4^2$,

(5) $2S_{11} + 5S_9 - S_7 = 6S_5^2$,

(6) $6S_{13} + 21S_{11} - 7S_9 + S_7 = 21S_6^2$,

(7) $3S_{15} + 14S_{13} - 7S_{11} + 2S_9 = 12S_7^2$,

(8) $10S_{17} + 60S_{15} - 42S_{13} + 20S_{11} - 3S_9 = 45S_8^2$,

(9) $2S_{19} + 15S_{17} - 14S_{15} + 10S_{13} - 3S_{11} = 10S_9^2$,

(10) $6S_{21} + 55S_{19} - 66S_{17} + 66S_{15} - 33S_{13} + 5S_{11} = 33S_{10}^2$.

Here we notice that the sum of numerical coefficients on the left is equal to the numerical coefficient on the right. For all the S's become 1 when we put $n = 1$.

2. Amigues gives, according to Lucas, in the *Nouv. Ann. de Math.* (2e Serie, t. X.) the formulae

$$4S_1^3 = 3S_5 + S_3, \quad 12S_2^3 = 16S_6 - 5S_4 + S_2.$$

Of these the first is evidently true, but not the latter. For, if we put $n = 2$, then we shall have

$$12S_2^3 = 1500, \text{ and } 16S_6 - 5S_4 + S_2 = 960.$$

Here we are required to find a substitute for this formula.

For this purpose we form a solid arranged with the general layer

$$\begin{array}{ccccc} r & 2r & 4r & \dots & nr \\ 2r & 4r & 6r & \dots & 2nr \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ nr & 2nr & 3nr & \dots & nnr \end{array}$$

The sum of this layer being

$$\sum_{k=1}^{n} krS_1 = rS_1^2,$$

$$\therefore \text{ the whole sum } = S_1^2 \sum_{r=1}^{n} r = S_1^3.$$

Next let us carry the summation in a different way. Thus we consider the whole sum distributed in districts whose general aspect is represented in the figure

$$\begin{array}{ccccc ccccc} p & 2p & 3p & \dots & pp & {\scriptstyle (p-1)p} & {\scriptstyle (p-2)p} & \dots & {\scriptstyle p} \\ 2p & 4p & 6p & \dots & 2pp & {\scriptstyle 2(p-1)p} & {\scriptstyle 2(p-2)p} & \dots & {\scriptstyle 2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ pp & 2pp & 3pp & \dots & ppp & {\scriptstyle p(p-1)p} & {\scriptstyle p(p-2)p} & \dots & {\scriptstyle pp} \end{array}$$

Here the large numerals represent those that are seen from one side, while the small numerals those that are arranged in a bi-side.

Now we have

$$\text{uppermost layer} = p\sum_{1}^{p} r + 2p\sum_{1}^{p} r + \dots + pp\sum_{1}^{p} r$$

$$= p\left(\frac{p(p+1)}{2}\right)^2.$$

$$\text{Sum of horizontal sides} = (p-r)p + 2(p-r)p + \dots + p(p-r)p$$

$$= 2p(p-r)\frac{p(p+1)}{2} - p^2(p-r)$$

for $r = 1, 2, \dots, (p-1)$. Hence the whole sum in this district will be

$$= \frac{p^3(p+1)^2}{4} + p^2(p+1)\frac{p(p-1)}{2} - \frac{p^3(p-1)}{2} = \frac{p^3}{4}(3p^2+1).$$

$$\therefore \sum_{p=1}^{n} \frac{p^3}{4}(3p^2+1) = S_1^3,$$

$$\therefore \sum_{p=1}^{n} (3p^5 + p^3) = 4 S_1^3, \text{ or } 3 S_5 + S_3 = 4 S_1^3,$$

which is the first of Amigues' formulae.

Again when the m^{th} powers of natural numbers are arranged as before, we get in like way

$$\sum_{p=1}^{n} \left\{ 3 p^m \sum_{r=1}^{p} r^m \sum_{k=1}^{p-1} k^m + p^{3m} \right\} = S_m^3.$$

In particular for $m = 2$, we have

$$3 p^2 \sum_{r=1}^{p} r^2 \sum_{k=1}^{p-1} k^2 + p^6 = \frac{p^4}{12}(4p^4 + 7p^2 + 1),$$

$$\therefore 4 S_8 + 7 S_6 + S_4 = 12 S_2^3,$$

which is the required relation that substitutes for Amigues' second formula.

Finally by making $m = 3, 4, 5$ we get

$$3 S_{11} + 10 S_9 + 3 S_7 = 16 S_3^3,$$

$$36 S_{14} + 195 S_{12} + 88 S_{10} - 20 S_8 + S_6 = 300 S_4^3,$$

$$4 S_{17} + 32 S_{15} + 21 S_{13} - 10 S_{11} + S_9 = 48 S_5^3.$$

II.

T. O., in Kyoto, publishes as a sequence to the results arrived at by T. Kariya a paper in the same journal of August, pp. 289—290. The following lines are a reproduction of the essay:

In general, S_m is an integral function of S_1 and it can be perfectly divided by S_1^2 or S_3, when m is odd; and when m is even, S_m can be expressed as the product of an integral function of S_1 and $2n+1$, and it can be divided by $S_1(2n+1)$ or S_2.

First to prove the proposition for the case of m even, we have in general

$$\frac{(x+1)^{k+1} x^{k+1} - x^{k+1} (x-1)^{k+1}}{2} = {}_{k+1}C_1\, x^{2k+1} + {}_{k+1}C_3\, x^{2k+3} + \ldots$$

In this expression by writing successively $n, n-1, \ldots, 2, 1$ for x, and adding the results together, we get

$$\frac{n^{k+1}(n+1)^{k+1}}{2} = 2^k S_1^{k+1} = {}_{k+1}C_1 \cdot S_{2k+1} + {}_{k+1}C_3 \cdot S_{2k-1} + \ldots,$$

where the last term is ${}_{k+1}C_k \cdot S_{k+2}$ or ${}_{k+1}C_{k+1} \cdot S_{k+1} = S_{k+1}$, according as k is odd or even.

For $k = 1, 2, 3, \ldots$ we have

$$\left.\begin{array}{l} {}_2C_1\, S_3 = 2\, S_1^{\,2}, \\ {}_3C_1\, S_5 + S_3 = 2^2 S_1^{\,3}, \\ {}_4C_1\, S_7 + {}_4C_3\, S_5 = 2^3 S_1^{\,4}, \ldots \end{array}\right\} \qquad \text{(A)},$$

whence successively follows that S_3, S_5, S_7, ... are divisible by $S_1^{\,2} = S_3$.

Next to prove for m even, we have in general

$$\frac{(x+1)^k (2x+1) x^k - x^k (2x-1)(x-1)^k}{2}$$

$$= \frac{(x+1)^{k+1} x^k - x^k (x-1)^{k+1}}{2} + \frac{(x+1)^k x^{k+1} - x^{k+1} (x-1)^k}{2}$$

$$= ({}_{k+1}C_1 + {}_kC_1)\, x^{2k} + ({}_{k+1}C_3 + {}_kC_3)\, x^{2k-2} + \ldots,$$

whence by writing $x = n, n-1, \ldots, 2, 1$, and adding together follows

$$\frac{(n+1)^k (2n+1)\, n^k}{2} = 2^{k-1} S_1^k (2n+1)$$

$$= ({}_{k+1}C_1 + {}_kC_1)\, S_{2k} + ({}_{k+1}C_3 + {}_kC_3)\, S_{2k-2} + \ldots,$$

where the last term is $({}_{k+1}C_k + {}_kC_k)\, S_{k+1}$ or $({}_{k+1}C_{k+1})\, S_k$ according as k is odd or even.

Here by writing successively $k = 1, 2, 3, \ldots$ we get

$$\left.\begin{array}{l} ({}_2C_1 + 1)\, S_2 = S_1 (2n+1), \\ ({}_3C_1 + {}_2C_1)\, S_4 + S_2 = 2\, S_1^{\,2} (2n+1), \\ ({}_4C_1 + {}_3C_1)\, S_6 + ({}_4C_3 + 1)\, S_4 = 2^2 S_1^{\,3} (2n+1), \ldots \end{array}\right\} \qquad \text{(B)},$$

whence we infer successively that S_2, S_4, ... are divisible by $S_1 (2n+1)$.

III.

K. Ogura's remark in the August number of the same journal, pp. 290—291.

The first of T. Kariya's formulae

$$4 S_1^{\,2} = 3 S_5 + S_3, \quad 12 S_2^{\,3} = 4 S_8 + 7 S_6 + S_4$$

is contained among the expressions (A) obtained by T. O., and the second may be easily derived in the following manner:

We take the identical relation

$$\frac{x^{k+1}}{2}\left\{(x+1)^{k+1}(2x+1)^{k+1}-(x-1)^{k+1}(2x-1)^{k+1}\right\}$$
$$=\sum_{\lambda=1}^{\lambda=k+1}\left\{\sum_{\mu=1}^{\mu=\lambda-1}(2^{\mu}+2^{2\lambda-\mu-1})_{k+1}C_{\mu}\cdot{}_{k+1}C_{2\lambda-\mu-1}\right\}x^{k+2\lambda},$$

where x, k, λ, μ are natural numbers.

Here x being put successively $=n$, $n-1$, ..., 2, 1, and the results being added together, we obtain

$$\frac{1}{2}[n(n+1)(2n+1)]^{k+1}=\frac{6^{k+1}}{2}S_2{}^{k+1}$$
$$=\sum_{\lambda=1}^{\lambda=k+1}\left\{\sum_{\mu=0}^{\mu=\lambda-1}(2^{\mu}+2^{2\lambda-\mu-1})_{k+1}C_{\mu}\cdot{}_{k+1}C_{2\lambda-\mu-1}\right\}S_{k+2\lambda},$$

which is our final result.

If specially we write $k=2$, we get T. Kariya's second formula

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N. YAMAMOTO, ON THE INTERPOLATION OF A FORMULA.[1])

In the continued product $1\cdot 2\cdot 3\ldots(n-1)n$ *the factors* $1\frac{1}{2}$, $2\frac{1}{2}$, ... *are to be intervened between its successive factors, and an expression for the resulting quantity is required.*

The result of such an interpolation can be easily arrived at by the application of the Γ function. But we propose to strike a different and simpler way in the following lines.

We designate the continued product of the first n natural numbers by

$$(\alpha)\qquad F(n)=1\cdot 2\cdot 3\ldots(n-1)n.$$

If n is replaced by $n+1$, it results

$$F(n+1)=1\cdot 2\cdot 3\ldots(n-1)n(n+1),$$

whose both members, divided by those of (α), afford

$$\frac{F(n+1)}{F(n)}=(n+1),$$

or

$$(\beta)\qquad F(n+1)=(n+1)F(n).$$

1) The Journal of the Society of Mathematics in Tokyo, Vol. 6, pp. 269—272, 1893.

The value of $F(1)$ is obviously 1, and successively putting $n=1$, 2, 3, . . ., we get

$$F(1)=1,$$
$$F(2)=2F(1)=1\times 2,$$
$$F(3)=3F(2)=1\times 2\times 3,$$
$$\cdots\cdots\cdots\cdots$$

Thus by means of the formula (β), the function $F(n)$ can be evaluated for whole number values of n.

When we make in (β) the substitutions $n=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, we obtain

$$F\left(\frac{3}{2}\right)=\frac{3}{2}F\left(\frac{1}{2}\right),$$
$$F\left(\frac{5}{2}\right)=\frac{5}{2}F\left(\frac{3}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}F\left(\frac{1}{2}\right),$$
$$F\left(\frac{7}{2}\right)=\frac{7}{2}F\left(\frac{5}{2}\right)=\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}F\left(\frac{1}{2}\right),$$
$$\cdots\cdots\cdots\cdots$$

and in general

$$(\gamma)\quad F\left(n-\frac{1}{2}\right)=F\left(\frac{2n-1}{2}\right)=\frac{2n-1}{2}\cdot\frac{2n-2}{2}\cdot\frac{2n-3}{2}\cdots\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}F\left(\frac{1}{2}\right).$$

We are therefore only required to make out the value of the quantity $F\left(\frac{1}{2}\right)$.

In (α) let us suppose that each of 1, 2, 3, . . . is decomposed into the product of two factors,

$$1=ab,\ 2=cd,\ldots,\ (n-1)=rs,\ n=tu,$$

so that we should have

$$F\left(\frac{1}{2}\right)=a,$$
$$F(1)=ab=1,$$
$$F\left(1\frac{1}{2}\right)=abc=1\times c,$$
$$F(2)=abcd=1\times 2,$$
$$F\left(2\frac{1}{2}\right)=abcde=1\times 2\times e,$$
$$\cdots\cdots\cdots\cdots$$
$$F(n-1)=abcd\ldots rs=1\cdot 2\ldots(n-1),$$
$$(\gamma')\qquad F\left(n-\frac{1}{2}\right)=abcd\ldots rst=1\cdot 2\ldots(n-t)t,$$
$$F(n)=abcd\ldots tu=1\cdot 2\ldots n.$$

From the combination of the two formulae (γ) and (γ'), it results

$$\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,F\left(\frac{1}{2}\right)=1\cdot 2\ldots(n-1)t,$$

whence we get

$$F\left(\frac{1}{2}\right)=\frac{2\cdot 4\cdot 6\ldots(2n-3)}{3\cdot 5\cdot 7\ldots(2n-1)}\cdot t,$$

or squared

$$\left\{F\left(\frac{1}{2}\right)\right\}^2=\frac{4\cdot 16\cdot 36\ldots(2n-3)^2}{9\cdot 25\cdot 49\ldots(2n-1)^2}\cdot t^2,$$

which, multiplied by

$$1=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1}\cdot\frac{1}{n},$$

becomes

$$\left\{F\left(\frac{1}{2}\right)\right\}^2=\frac{8}{9}\cdot\frac{24}{25}\cdot\frac{48}{49}\cdots\frac{(2n-1)^2-1}{(2n-1)^2}\cdot\frac{t^2}{n}.$$

But $tu=n$, $\therefore\ \frac{t}{n}=\frac{1}{u}$, and $\frac{t^2}{n}=\frac{t}{u}\cdot$

We have therefore

$$(\delta)\qquad \left\{F\left(\frac{1}{2}\right)\right\}^2=\frac{8}{9}\cdot\frac{24}{25}\cdot\frac{48}{49}\cdots\frac{(2n-1)^2-1}{(2n-1)^2}\cdot\frac{t}{u}.$$

When n is a sufficiently large number, the quotient of the last two factors in (α), namely, $\frac{n-1}{n}$, has a value that approaches to unity and it becomes at last $=1$ in the limit for which $n=\infty$. It ought the same circumstance to reign over the formula (δ). We should have therefore $\frac{t}{u}=1$ in the limit $n=\infty$. Consequently it follows from (δ), when we make $n=\infty$ in it, that

$$\left\{F\left(\frac{1}{2}\right)\right\}^2=\frac{8}{9}\cdot\frac{24}{25}\cdot\frac{48}{49}\cdots\qquad\text{ad. inf.}$$

$$=\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\cdots,$$

which, we know, is equal to $\frac{1}{4}\pi$.

$$\therefore\ F\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}.$$

We get therefore the required formula in the form

$$F\left(\frac{m}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{m}{2}\sqrt{\pi}.$$

K. MORI, A STUDY ON THE CUBIC EQUATION.[1])

In the following lines we give a method for the solution of the cubic equation, that is different from that of Cardan. If our process will not be very convenient for practical purposes, yet it is not without some theoretical interests.

1. In the equation

$$f(x) \equiv x^3 + Ax^2 + Bx + C = 0,$$

let it be supposed that there exists between its coefficients a relation of the form

$$f(1)\, f''(1) : f'(1)^2 = 2 : 3.$$

In this case a simple transformation enables us to solve it. For the substitution, $x = \frac{z}{y} + 1$, transforms our equation into

$$\frac{z^3}{y^3} + \frac{1}{2} f''(1) \frac{z^2}{y^2} + f'(1) \frac{z}{y} + f(1) = 0,$$

or

$$y^3 + \frac{f'(1)\, z}{f(1)} \cdot y^2 + \frac{f''(1)\, z^2}{2 f(1)} \cdot y + \frac{z^3}{f(1)} = 0.$$

Here z may have any arbitrary value. Let it be determined so as the coefficient of y^2 should become 3.

Then making use of our assumed condition, $3 f(1)\, f''(1) = 2 f'(1)^2$, we have

$$\frac{f''(1)\, z^2}{2 f(1)} = \frac{1}{3} \left\{ \frac{f'(1)\, z}{f(1)} \right\}^2 = \frac{1}{3} \times 3^2 = 3.$$

Hence our equation becomes

$$y^3 + 3y^2 + 3y + \frac{27 f(1)^2}{f'(1)^3} = 0,$$

or

$$(y+1)^3 = 1 - \frac{27 f(1)^2}{f'(1)^3},$$

which is easy to solve.

2. Now any equation of the third degree can be brought into the form just considered.

For let

$$f(x) \equiv x^3 + px^2 + qx + r = 0$$

be a cubic equation in its most general form. The substitution $x = y/m$, where m is an yet undetermined constant, transforms it into

$$y^3 + pmy^2 + qm^2 y + rm^3 = 0,$$

1) Journal of Phys. School, Vol. 3, pp. 276—279. 1894.

in which we may impose on m a value that will arise from the relation $f(1)f''(1):f'(1)^2=2:3$, that is, from the equation

$$(q^2-3pr)m^2+(pq-qr)m+p^2-3q=0.$$

The value of m determined from this equation being substituted in the equation in y, it assumes a form as required.

3. From the above consideration our process appears to be applicable to any equation of the third degree. When the value of m is easily obtainable in a simple form, the process will present no difficulty in practice; but if m becomes imaginary, there will arise some practical inconveniencies.

For a real value of m there will correspond a real root in x, the remaining two being imaginary.

When m has a complex value, the equation being in the irreducible form, it will have its three roots all real.

K. MORI, ON THE SOLUTION OF EQUATIONS IN ONE UNKNOWN QUANTITY.[1])

1. *To find a function $f(x)$ that satisfies the relation*

$$f(x)^n+xf(x)=1,$$

we assume for $f(x)$

$$f(x)=A_0+A_1x+A_2x^2+\ldots,$$

and substituting it in the equation, we obtain

$$(A_0+A_1x+A_2x^2+\ldots)^n+x(A_0+A_1x+A_2x^2+\ldots)=1,$$

whence the comparison of coefficients leads us to the identities

$$A_0{}^n=1,$$
$$nA_0{}^{n-1}A_1+A_0=0,$$
$$nA_0{}^{n-1}A_2+\frac{n(n-1)}{2!}A_0{}^{n-2}A_1{}^2+A_1=0,$$
$$nA_0{}^{n-1}A_3+2\frac{n(n-1)}{2!}A_0{}^{n-2}A_1A_2+\frac{n(n-1)(n-2)}{3!}A_0{}^{n-3}A_1{}^3+A_2=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

or what is the same thing,

1) The present paper is compiled from the results of two papers published by K. Mori in the Journal of the Physics School in Tokyo with the titles *On solving the equation of the fifth degree* (April, 1903), and *On solving the equation with one unknown quantity,* which appeared in several parts from May to December of 1903.

$$A_0 = 1, \quad A_1 = -\frac{1}{n}, \quad A_2 = -\frac{n-3}{2!\,n^2},$$

$$A_3 = -\frac{(n-4)(2n-4)}{3!\,n^3}, \quad A_4 = -\frac{(n-5)(2n-5)(3n-5)}{4!\,n^4}, \ldots$$

We have therefore for the expression of $f(x)$

$$f(x) = 1 - \frac{x}{n} - \frac{n-3}{2!\,n^2}x^2 - \frac{(n-4)(2n-4)}{3!\,n^3}x^3 - \frac{(n-5)(2n-5)(3n-5)}{4!\,n^4}x^4 - \ldots$$

2. *This expansion is directly applicable for the determination of another function defined by the equation*

$$\varphi(x)^n + x\varphi(x) = t^n.$$

For writing $\varphi(x) = t f(x)$, this equation reduces to

$$f(x)^n + x t^{1-n} f(x) = 1.$$

We get therefore by last article

$$f(x) = 1 - \frac{x}{nt^{n-1}} - \frac{(n-3)x^2}{2!\,n^2 t^{2(n-1)}} - \frac{(n-4)(2n-4)x^3}{3!\,n^3 t^{3(n-1)}} - \frac{(n-5)(2n-5)(3n-5)x^4}{4!\,n^4 t^{4(n-1)}} - \ldots,$$

or

$$\varphi(x) = t - \frac{x}{nt^{n-2}} - \frac{(n-3)x^2}{2!\,n^2 t^{2n-3}} - \frac{(n-4)(2n-4)x^3}{3!\,n^3 t^{3n-4}} - \frac{(n-5)(2n-5)(3n-5)x^4}{4!\,n^4 t^{4n-5}} - \ldots$$

But if we make in the binomial expansion of

$$\frac{t}{\alpha}(1 + xt^{1-n})^{-\frac{\alpha}{n}}$$

the value of α in each term separately equal to the index of t in that term, we shall evidently arrive at the same series as the above. A series of this kind will be conveniently denoted by

$$S\frac{t}{\alpha}(1 + xt^{1-n})^{-\frac{\alpha}{n}},$$

where the symbol S indicates the said property or operation.

We have thus

$$\varphi(x) = S\frac{t}{\alpha}(1 + xt^{1-n})^{-\frac{\alpha}{n}}.$$

We designate the function S as a *root function.*

3. From what has been treated in last article it is evident that *one of the roots of the equation*

$$x^n + a\,x = t^n$$

can be expressed by

$$x = S\,\frac{t}{\alpha}\,(1 + a\,t^{1-n})^{-\frac{\alpha}{n}}.$$

4. Next in the equation,

$$x^m + a\,x^n = t^{\frac{m}{n}},$$

make the substitution $x^n = y$, and it will be reduced to

$$y^{\frac{m}{n}} + a\,y = t^{\frac{m}{n}},$$

so that we get by last article

$$y = S\,\frac{t}{\alpha}\left(1 + a\,t^{1-\frac{m}{n}}\right)^{-\frac{n\alpha}{m}},$$

and hence

$$x = \left\{ S\,\frac{t}{\alpha}\left(1 + a\,t^{1-\frac{m}{n}}\right)^{-\frac{n\alpha}{n}} \right\}^{\frac{1}{n}}.$$

In cases where the expanded series are found divergent, the series in the expansion of

$$x = \left\{ S\,\frac{t_1}{\alpha}\left(1 + a^{-1}t_1^{\,1-\frac{n}{m}}\right)^{-\frac{m\alpha}{n}} \right\}^{\frac{1}{m}}$$

will be necessarily convergent.

From the result in this article the solution of the quintic equation in its reduced forms can be effected in an easy manner.[1])

5. *The root of the equation*

$$x^m + a\,x^n = t^m$$

will find its expression in

$$x = S\,\frac{t}{\alpha}\,(1 + a\,t^{n-m})^{-\frac{\alpha}{m}},$$

a result that will be arrived at in a similar manner as in the case for $x^m + a\,x = t^m$. (As to the contents of this article no further mention is made by the author.)

6. *To obtain the m roots of the equation*

$$x^m + a\,x^n = t^m,$$

1) The above is the way followed by the author in his paper on the quintic equation, where he gives the expressions for the roots.

it is only necessary in place of t to take $t\,\omega^r$ for $r = 1, 2, \ldots, m$, where ω denotes one of the complex roots of $x^m = 1$. The general expression for the root is therefore

$$x + S\frac{t\,\omega^r}{\alpha}\{1 + a\,t^{n-m}\omega^{rn}\}^{-\frac{\alpha}{m}}.$$

7. *The equation* $x^m + a\,x^n = t^m$ reduces, through the substitution of $x = y^\nu$, to the form

$$y^{\nu m} + a\,y^{\nu n} + \left(t^{\frac{1}{\nu}}\right)^{\nu m} = t_1^{\nu m}, \text{ say,}$$

whence we have

$$y = S\frac{t_1}{\alpha}(1 + a\,t_1^{\nu n - \nu m})^{-\frac{\alpha}{\nu m}},$$

and hence

$$x = \left\{S\frac{t_1}{\alpha}(1 + a\,t_1^{\nu n - \nu m})^{-\frac{\alpha}{\nu m}}\right\}^\nu,$$

or the S function being expanded,

$$x = t\left\{1 - \frac{a}{\nu m t^{m-n}} - \frac{(\nu m - 2\,\nu n - 1)\,a^2}{2!\ \nu^2 m^2 t^{2(m-n)}}\right.$$
$$\left. - \frac{(\nu m - 3\,\nu n - 1)\,(2\,\nu m - 3\,\nu n - 1)\,a^3}{3!\ \nu^2 m^3 t^{3(m-n)}} - \ldots\right\}^\nu,$$

where ν may have any value we please.

Writing $\frac{1}{\nu}$ for ν we have

$$x = t\left\{1 - \frac{\nu a}{m t^{m-n}} - \frac{\nu\,(m - 2\,n - \nu)\,a^2}{2!\ m^2 t^{2(m-n)}}\right.$$
$$\left. - \frac{\nu\,(m - 3\,n - \nu)\,(2\,m - 3\,n - \nu)\,a^3}{3!\ m^3 t^{3(m-n)}} - \ldots\right\}^{\frac{1}{\nu}}.$$

From this we get

$$\frac{x^\nu - 1}{\nu} = \frac{t^\nu - 1}{\nu} - t^\nu\left\{\frac{a}{m\,t^{m-n}} + \frac{(m - 2\,n - \nu)\,a^2}{2!\ m^2 t^{2(m-n)}}\right.$$
$$\left. + \frac{(m - 3\,n - \nu)\,(2\,m - 3\,n - \nu)\,a^3}{3!\ m^3 t^{3(m-n)}} + \ldots\right\}.$$

Now we go over to the limit $\nu = 0$, when it results

$$\log x = \log t - \left\{\frac{a}{m\,t^{m-n}} + \frac{(m - 2\,n)\,a^2}{2!\ m^2 t^{2(m-n)}} + \frac{(m - 3\,n)\,(2\,m - 3\,n)\,a^3}{3!\ m^3 t^{3(m-n)}} + \ldots\right\},$$

that is,

$$\log x = \log t + S\frac{1}{\alpha}\left\{(1 + a\,t^{n-m})^{-\frac{\alpha}{m}} - 1\right\},$$

and hence

$$x = t\,e^{S\frac{1}{\alpha}\left\{(1+at^{n-m})^{-\frac{\alpha}{m}}-1\right\}}.$$

This result is capable also to be written in the form

$$x = t\left[\cos h\ S\frac{1}{\alpha}\left\{(1 + a\,t^{n-m})^{-\frac{\alpha}{m}} - 1\right\}\right.$$

$$\left. + \ \sin h\ S\frac{1}{\alpha}\left\{(1 + a\,t^{n-m})^{-\frac{\alpha}{m}} - 1\right\}\right].$$

8. *The process just described may be advantageously applied to the solution of some transcendental equations.*

Thus we take the well-known *equation*

(1) $$e^x = 1 + y.$$

Here the value of x is to be expanded in terms of y.

For this purpose we first take the equation

(2) $$\left(\frac{x+n}{n}\right)^n = 1 + y\left(\frac{x+n}{n}\right),$$

and solve it for $\frac{x+n}{n}$, when our notation will give

$$\frac{x+n}{n} = S\frac{t}{\alpha}(1 - y\,t^{1-n})^{-\frac{\alpha}{n}},$$

where t is to be put ultimately equal to unity, after the indicated operation is effected.

It follows then

$$x = n\,S\frac{t}{\alpha}(1 - y\,t^{1-n})^{-\frac{\alpha}{n}} - n,$$

or, the expansion being carried out and t being replaced by its ultimate value,

(3) $$x = y + \frac{3-n}{2!\ n}y^2 + \frac{(4-n)(4-2n)}{3!\ n^2}y^3 + \ldots$$

Now, in going to the limit, for which n indefinitely increases, the equation (2) reduces to (1), as will be easily seen. We deduce therefore the value of x, that satisfies (1), from the formula (3), by making $n = \infty$ in it. The result is

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$$

9. A second way also grounded on our method may serve for the solution of the same problem. We form the identity

$$x^n = (1+y)\,x^n - y\,x^{n-1}\,x,$$

or as we choose to write

$$(x)^n = (1+y)\,x^n - y\,x^{n-1}\,(x),$$

and we shall consider (x) as the unknown quantity, while the x not enclosed in brackets will be treated as were a constant for a while. The root of this equation, thus considered, will assume, after expansion by our way, the form

$$(x) = (1+y)^{\frac{1}{n}}\,x\left\{1 - \frac{y}{n(1+y)^{\frac{n-1}{n}}} + \frac{(3-n)\,y^2}{2!\,n^2(1+y)^{\frac{2(n-1)}{n}}} - \frac{(4-n)(4-2n)\,y^3}{3!\,n^3(1+y)^{\frac{3(n-1)}{n}}} + \dots\right\}.$$

From this formula the factor x being cancelled out, we get after transposition

$$\frac{n\left\{(1+y)^{\frac{1}{n}} - 1\right\}}{(1+y)^{\frac{1}{n}}} = \frac{y}{(1+y)^{\frac{n-1}{n}}} - \frac{\left(\frac{3}{n}-1\right)y^2}{2!\,(1+y)^{\frac{2(n-1)}{n}}} + \dots$$

We now go to the limit $n = \infty$. Then since

$$\lim_{n=\infty} n\left\{(1+y)^{\frac{1}{n}} - 1\right\} = \log(1+y),$$

we get

$$\log(1+y) = \frac{y}{1+y} + \frac{y^2}{2\,(1+y)^2} + \frac{y^3}{3\,(1+y)^3} + \dots$$

10. *To solve the equation*

(1) $$x = te^{ax^n},$$

we start with the equation

(2) $$x^m = t^m + amx^n,$$

whose solution is, as we have already seen,

(3) $$x = S\,\frac{t}{\alpha}\,(1 - amt^{n-m})^{-\frac{\alpha}{m}}.$$

We can rewrite the equation (2) in the form

$$\frac{x^m - 1}{m} = \frac{t^m - 1}{m} + ax^n,$$

which, in the limit $m = 0$, reduces to

$$\log x = \log t + ax^n,$$

an equation that expresses the same thing as (1).

Hence the required solution of (1) will be obtained from (3), by making $m = 0$ therein. Therefore we have

$$x = t + at^{n+1} + \frac{(1+2n)a^2t^{2n+1}}{2!} + \frac{(1+3n)^2a^3t^{3n+1}}{3!} + \ldots$$

11. If we put $t = 0$ in the equation of last article, and change n into $-n$, then we get *the equation*

$$x^{x^n} = e^a;$$

at the same time the solution assumes the form

$$x = 1 + a + \frac{(1-2n)a^2}{2!} + \frac{(1-3n)^2a^3}{3!} + \ldots$$

12. *To solve the equation*

(1) $$x = \log t + ae^{nx}.$$

The solution of

(2) $$e^{mx} = t^m + ame^{nx}$$

will be obtained in the form

$$e^x = te^{S\frac{1}{\alpha}\left\{(1-amt^{n-m})^{-\frac{\alpha}{m}}-1\right\}},$$

or

(3) $$x = \log t + S\frac{1}{\alpha}\left\{(1-amt^{n-m})^{-\frac{\alpha}{m}} - 1\right\}.$$

But we can write (2) in the form

$$\frac{e^{mx}-1}{m} = \frac{t^m - 1}{m} + ae^{nx},$$

which reduces to (1) in the limit $m = 0$.

The required solution of (1) will therefore be obtained from (3) by making $m = 0$. The result is

$$x = \log t + at^n + na^2t^{2n} + \frac{3n^2a^3t^{3n}}{2!} + \frac{4n^3a^4t^{4n}}{3!} + \ldots$$

13. *Expansions of* $\sin x$ *and* $\cos x$.

By de Moivre's formulae we have

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$

and

$$(\cos x + i \sin x)^{-n} = \cos nx - i \sin nx.$$

By subtraction we get, after a reduction,

$$(\cos x + i \sin x)^{2n} - 2 i \sin nx (\cos x + i \sin x)^n = 1 = t^{2n}.$$

The quantity $(\cos x + i \sin x)$ being considered as the unknown, we obtain

$$\cos x + i \sin x = S \frac{t}{\alpha} (1 - 2i \sin nx \cdot t^{-n})^{-\frac{\alpha}{2n}}.$$

The expansion on the right-hand side being effected and the substitution $t = 1$ being made, we have

$$\cos x + i \sin x = 1 + \frac{i \sin nx}{n} - \frac{\sin^2 nx}{2!\, n^2} - \frac{i (1 - n^2) \sin^3 nx}{3!\, n^2} + \frac{(1 - 2^2 n^2) \sin^4 nx}{4!\, n^4} + \ldots$$

The separation of real and imaginary parts gives

$$\cos x = 1 - \frac{\sin^2 nx}{2!\, n^2} + \frac{(1 - 2^2 n^2) \sin^4 nx}{4!\, n^4} - \frac{(1 - 2^2 n^2)(1 - 4^2 n^2) \sin^6 nx}{6!\, n^6} + \ldots,$$

and

$$\sin x = \frac{\sin nx}{n} - \frac{(1 - n^2) \sin^3 nx}{3!\, n^3} + \frac{(1 - n^2)(1 - 3^2 n^2) \sin^5 nx}{5!\, n^5} - \ldots$$

If we make $n = 0$ in these formulae, we shall then have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{6!} - \ldots,$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

When the substitutions $x = mx$, $n = \frac{1}{m}$ are made in the former expressions, we shall have

$$\cos mx = 1 - \frac{m^2 \sin^2 x}{2!} + \frac{m^2 (m^2 - 2^2) \sin^4 x}{4!} - \frac{m^2 (m^2 - 2^2)(m^2 - 4^2) \sin^6 x}{6!} + \ldots,$$

$$\sin mx = m \sin x - \frac{m^2 (m^2 - 1^2) \sin^3 x}{3!} + \frac{m^2 (m^2 - 1^2)(m^2 - 3^2) \sin^5 x}{5!} - \ldots$$

Thus we have arrived at these longst known formulae through our own way.

14. In the identity

$$e^{2xi} - 2i \sin x \cdot e^{xi} = 1 = t^{2xi},$$

if we consider e temporarily as the unknown quantity, then we have

$$e = S\frac{t}{\alpha}(1 - 2i\sin x \cdot t^{-xi})^{-\frac{\alpha}{2xi}},$$

or in its expanded form, t being replaced by its value,

$$e = 1 + \frac{\sin x}{x} + \frac{\sin^2 x}{2!\,x^2} + \frac{(1+x^2)\sin^3 x}{3!\,x^3} + \frac{(1+2^2x^2)\sin^4 x}{4!\,x^4}$$
$$+ \frac{(1+x^2)(1+3^2x^2)\sin^5 x}{5!\,x^5} + \dots$$

15. In the last place we proceed to *the establishment of our process for the general case of an algebraical equation.*

The root of the equation

$$x^n + a_1 x^{n-1} = t^n$$

is evidently

$$x = S\frac{t}{\alpha}(1 + a_1 t^{-1})^{-\frac{\alpha}{n}},$$

or expanded

$$x = t - \frac{a_1}{n} - \frac{(1-n)a_1^{\,2}}{2!\,n^2 t} - \frac{(2-n)(2-2n)a_1^{\,3}}{3!\,n^3 t^2} - \dots$$

From this we guess that the root of the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} = t^n$$

is capable of a similar expansion. We therefore assume for x

$$x = t + A + A_1 t^{-1} + A_2 t^{-2} + \dots$$

Thus we shall have the relation

$$(t + A + A_1 t^{-1} + A_2 t^{-2} + \dots)^n$$
$$+ a_1(t + A + A_1 t^{-1} + A_2 t^{-2} + \dots)^{n-1}$$
$$+ a_2(t + A + A_1 t^{-1} + A_2 t^{-2} + \dots)^{n-2} = t^n.$$

A comparison of coefficients in this relation gives for the A's the values on calculation:

$$A = -\frac{a_1}{n},$$

$$A_1 = -\frac{(1-n)a_1^{\,2} + 2!\,n a_2}{2!\,n^2},$$

$$A_2 = -\frac{(2-n)(2-2n)a_1^{\,3} + 3!\,n(2-n)a_1 a_2}{3!\,n^3},$$

$$A_3 = -\frac{(3-n)(3-2n)(3-3n)a_1^{\,4} + \frac{4!}{2!}n(3-n)(3-2n)a_1^{\,2}a_2 + \frac{4!}{2!}n^2(3-n)a_2^{\,2}}{4!\,n^4},$$

. . . .

We have therefore

$$x = t - \frac{a_1}{n} - \frac{(1-n)a_1{}^2 + 2!\,n a_2}{2!\,n^2 t} - \frac{(2-n)(2-2n)a_1{}^3 + 3!\,n(2-n)a_1 a_2}{3!\,n^3 t^2} - \ldots$$

But the S function

$$S\frac{t}{\alpha}(1 + a_1 t^{-1} + a_2 t^{-2})^{-\frac{\alpha}{n}}$$

will be found on expansion to be identical with the above value of x. We have therefore

$$x = S\frac{t}{\alpha}(1 + a_1 t^{-1} + a_2 t^{-2})^{-\frac{\alpha}{n}}$$

as the root of our equation.

A repetition of the same reasoning will give step by step the roots of the equations that contain 4, 5, 6, ... of the highest terms in x; and finally for the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x = t^n$$

we arrive at the solution in the form

$$x = S\frac{t}{\alpha}(1 + a_1 t^{-1} + a_2 t^{-2} + \ldots + a_{n-1} t^{1-n})^{-\frac{\alpha}{n}},$$

or, as may be written also,

$$x = S\frac{t^{1+\alpha}}{\alpha}(t^n + a_1 t^{n-1} + a_2 t^{n-2} + \ldots + a_{n-1} t)^{-\frac{\alpha}{n}}.$$

By replacing t by $t\,\omega^r$, where ω denotes one of the imaginary roots of $x^n - 1 = 0$, and where r stands for $r = 1, 2, \ldots, n$, the expression above obtained may be made to represent the n roots of the equation in turn.

K. MORI, ON A FUNCTIONAL FORM THAT SERVES THE SOLUTION OF AN EQUATION.[1]

§ 1. We consider a function of the form

$$\{1 + f(x)\}^{\delta}\,\varphi(x),$$

where $f(x)$ and $\varphi(x)$ are any functions of x, and where the symbol δ

1) The Journal of Tokyo Physics School, Vol. 15, pp. 331—335, Aug. 1906. It is there stated by the author that the present article is an abridged reproduction from a newly written work with the title of *The Sōgō-Gaku*, that is intended for publication.

stands for the differential operator $x\frac{d}{dx}$, whose operation we suppose to be carried in such a manner as follows:

$$\{1+f(x)\}^{\delta}\varphi(x) = \Big\{1+\delta f(x)+\frac{\delta(\delta-1)}{2!}f^2(x)$$
$$+\frac{\delta(\delta-1)(\delta-2)}{3!}f^3(x)+\dots$$
$$+\frac{\delta(\delta-1)(\delta-2)\dots(\delta-r+1)}{r!}f^r(x)+\dots\Big\}\varphi(x)$$
$$=\varphi(x)+x\frac{d}{dx}\{f(x)\varphi(x)\}+\frac{1}{2!}x^2\frac{d^2}{dx^2}\{f^2(x)\varphi(x)\}$$
$$+\frac{1}{3!}x^3\frac{d^3}{dx^3}\{f^3(x)\varphi(x)\}+\dots$$
$$+\frac{1}{r!}x^r\frac{d^r}{dx^r}\{f^r(x)\varphi(x)\}+\dots$$

We designate the last member of the above equality as the *value* of the first member.

§ 2. We adopt for the value of $\{1+f(x)\}^{\delta}\varphi(x)$ the notation of the form

$$S[\{1+f(x)\}^{\delta}\varphi(x)].$$

Since this value is a function of x alone, as will be seen from § 1, we can put

$$S[\{1+f(x)\}^{\delta}\varphi(x)]=\psi(x).$$

In like manner the symbol $S[f(\delta\cdot x)]$ will be employed for the result obtained after the operation of δ in $f(\delta\cdot x)$.

§ 3. *To evaluate*

$$S[\{1+f(x)\}^{-\frac{\delta}{n}}\varphi(x)],$$

we put

$$x=y^{-\frac{1}{n}} \text{ or } x^{-n}=y,$$

and differentiating both sides we have

$$-n x^{-n-1}dx=dy,$$

$$\therefore\ y\frac{d}{dx}=x^{-n}\cdot\frac{d}{-nx^{-n-1}}=-\frac{1}{n}x\frac{d}{dx}=-\frac{\delta}{n},$$

which we denote by δ_1.

We get then

$$S\Big[\{1+f(x)\}^{-\frac{\delta}{n}}\varphi(x)\Big]=S\Big[\Big\{1+f\Big(y^{-\frac{1}{n}}\Big)\Big\}^{\delta_1}\varphi\Big(y^{-\frac{1}{n}}\Big)\Big].$$

The value of this formula will be found by § 1.

§ 4. We have as is well known

$$\delta^n x^p = p^n x^p \text{ and } (\delta + m)^n x^p = (p + m)^n x^p.$$

These formulae, denoted according to the notation in § 2, take the forms

$$S[\delta^n x^p] = p^n x^p \text{ and } S[(\delta + m)^n x^p] = (p + m)^n x^p.$$

§ 5. $x^m S[\delta^n x^p] = S[(\delta - m)^n x^{p+m}]$,

for by § 4 we have

$$x^m S[\delta^n x^p] = x^m \cdot p^n x^p = p^n x^{p+m},$$

and

$$S[(\delta - m)^n x^{p+m}] = (p + m - m)^n x^{p+m} = p^n x^{p+m}.$$

§ 6. $x^m S[f(\delta \cdot x)] = S[f\{(\delta - m) x\} x^m]$.

For, if we put

$$\text{(I)} \qquad f(\delta \cdot x) = A\, \delta^n x^p + A_1 \delta^{n_1} x^{p_1} + A_2 \delta^{n_2} x^{p_2} + \ldots,$$

the general term in the expansion of the left-hand member of the formula to be established will be, by § 5,

$$x^m S[A_r \delta^{n_r} x^{p_r}] = S[A_r (\delta - m)^{n_r} x^{p_r + m}],$$

so that,

$$\begin{aligned} x^m S[f(\delta \cdot x)] = S[(\delta - m)^n x^{p+m} &+ A_1 (\delta - m)^{n_1} x^{p_1 + m} \\ &+ A_2 (\delta - m)^{n_2} x^{p_2 + m} + \ldots \\ &+ A_r (\delta - m)^{n_r} x^{p_r + m} + \ldots] \\ = S[\{A (\delta - m)^n x^p &+ A_1 (\delta - m)^{n_1} x^{p_1} \\ &+ A_2 (\delta - m)^{n_2} x^{p_2} + \ldots \\ &+ A_r (\delta - m)^{n_r} x^{p_r} + \ldots\} x^m]. \end{aligned}$$

The quantity within the crooked brackets is the same as $f(\delta \cdot x)$ by (I) with the difference that $\delta - m$ should be written in place of δ.

Hence our formula follows.

§ 7. If

$$\text{(I)} \qquad \varphi(x) \Big[1 + f\{\varphi(x)\}\Big]^{-1} = x,$$

we shall have

$$\text{(II)} \qquad S[\{1 + f(x)\}^\delta F(x)] = F\{\varphi(x)\} S[\{1 + f(x)\}^\delta].$$

For according to § 2 we may make the replacement

$$S[\{1 + f(x)\}^\delta] = \psi(x),$$

and by multiplying with the factor x^n, we get

$$S[\{1+f(x)\}^\delta \{1+f(x)\}^{-n} x^n] = x^n \psi(x),$$

which, multiplied with $\frac{F^{(n)}(y)}{n!}$, gives

$$S\left[\{1+f(x)\}^\delta \left\{\frac{F^{(n)}(y)}{n!}\{1+f(x)\}^{-n} x^n\right\}\right] = \frac{F^{(n)}(y)}{n!} x^n \psi(x),$$

where $F(y)$ denotes an arbitrary function of y, which is independent of x.

In the above formula, taking for n successively the values $0, 1, 2, 3, \ldots, r, \ldots$, we get

$$S[\{1+f(x)\}^\delta F(y)] = F(y)\psi(x),$$
$$S[\{1+f(x)\}^\delta F'(y)\{1+f(x)\}^{-1} x] = F'(y) x \psi(x),$$
$$S\left[\{1+f(x)\}^\delta \frac{F''(y)}{2!}\{1+f(x)\}^{-2} x^2\right] = \frac{F''(y)}{2!} x^2 \psi(x),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S\left[\{1+f(x)\}^\delta \frac{F^{(r)}(y)}{r!}\{1+f(x)\}^{-r} x^r\right] = \frac{F^{(r)}(y)}{r!} x^r \psi(x),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

These expressions being added together we obtain by Taylor's formula

$$S\left[\{1+f(x)\}^\delta F\Big(y+\{1+f(x)\}^{-1}x\Big)\right] = F(y+x)\psi(x).$$

Making the substitution $y=0$, we have

$$S\left[\{1+f(x)\}^\delta F\Big(\{1+f(x)\}^{-1}x\Big)\right] = F(x)\psi(x) = F(x)\, S[\{1+f(x)\}^\delta].$$

But $F(x)$ is an arbitrary function of x, so that we can write

$$F(x) = F\{\varphi(x)\},$$

and so also

$$F\Big(\{1+f(x)\}^{-1}x\Big) = F\Big([1+f\{\varphi(x)\}]^{-1}\varphi(x)\Big).$$

These values being substituted the above formula transforms into

$$S\left[\{1+f(x)\}^\delta F\Big([1+f\{\varphi(x)\}]^{-1}\varphi(x)\Big)\right] = F\{\varphi(x)\}\, S[\{1+f(x)\}^\delta].$$

But $\varphi(x)$ is also an arbitrary function, so we may choose for it the existence of the relation (I). We have then

$$S[\{1+f(x)\}^\delta F(x)] = F\{\varphi(x)\}\, S[\{1+\varphi(x)\}^\delta],$$

which is the formula that was to be established.

§ 8. If

$$\text{(I)} \qquad \varphi^n(x)\,[1+f\{\varphi(x)\}] = x^n,$$

then

$$\text{(II)} \qquad S\Big[\{1+f(x)\}^{-\frac{\delta}{n}} F(x)\Big] = F\{\varphi(x)\}\, S\Big[\{1+f(x)\}^{-\frac{\delta}{n}}\Big].$$

For $f(x)$ in last article is an arbitrary function, so that we can write for it

$$f(x) = \{1+f_1(x)\}^{-\frac{1}{n}} - 1,$$

without introducing any restriction whatever.

In this case we get

$$\text{(III)} \qquad S\Big[\{1+f(x)\}^{\delta}\Big] = S\Big[\{1+f_1(x)\}^{-\frac{\delta}{n}}\Big].$$

At the same time (I) in § 7 becomes

$$[1+f_1\{\varphi(x)\}]^{\frac{1}{n}}\varphi(x) = x,$$

or

$$\text{(IV)} \qquad \varphi^n(x)\,[1+f_1\{\varphi(x)\}] = x^n,$$

and (II) in § 7 becomes by (III)

$$\text{(V)} \qquad S\Big[\{1+f_1(x)\}^{-\frac{\delta}{n}} F(x)\Big] = F\{\varphi(x)\}\, S\Big[\{1+f_1(x)\}^{-\frac{\delta}{n}}\Big].$$

When therefore we replace $f_1(x)$ by $f(x)$, we arrive at once at our required result.

§ 9. The formula of last article adapts itself to a wide scope of applications. As an example we apply it to the solution of an equation in one unknown quantity.

In § 8 we may write $x=t$, $\varphi(x)=x$, and thus we have

$$\text{(I)} \qquad x^n\{1+f(x)\} = t^n,$$

$$\text{(II)} \qquad S\Big[\{1+f(t)\}^{-\frac{\delta}{n}} F(t)\Big] = F(x)\, S\Big[\{1+f(t)\}^{-\frac{\delta}{n}}\Big].$$

From (II) we get

$$\text{(III)} \qquad F(x) = \frac{S\Big[\{1+f(t)\}^{-\frac{\delta}{n}} F(t)\Big]}{S\Big[\{1+f(t)\}^{-\frac{\delta}{n}}\Big]},$$

where $\delta = t\dfrac{d}{dt}$.

The formula (III) represents an arbitrary function of the root x of the equation (I), and the numerator and denominator can be evaluated by § 3.

If we put $F(x) = x$, we get

$$x = \frac{S\left[\{1+f(t)\}^{-\frac{\delta}{n}} t\right]}{S\left[\{1+f(t)\}^{-\frac{\delta}{n}}\right]},$$

which gives the general solution of the equation (I).

U. FUJIMAKI, ON THE SUM OF A HARMONICAL PROGRESSION.

I.

In *the Journal of the Tokyo Physics School, Vol. 5, pp. 235—239 and 265—268, August and September, 1896,* U. Fujimaki gives a paper *On the limits of the sum of a harmonical progression.*

There he begins with these words:

"As we believe, we cannot, at the present state of knowledge, give an expression for the sum of any number of terms that constitute a harmonical progression. But as to the limits between which such a sum would lie, I once learned from a friend of mine, that those limits could be calculated in some way or other. Instigated by my friend's words, I set myself on a study of this subject and have come to a result tolerably agreeable."

The author then proceeds essentially as in the following lines:

If we put

$$\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \ldots + \frac{1}{a+(n-1.b} = S_n;$$

then we have

$$(1) \qquad \frac{b}{a} + \frac{b}{a+b} + \frac{b}{a+2b} + \ldots + \frac{b}{a+(n-1)b} = bS_n.$$

By subtracting both sides of (1) from n and rearranging the result, we get

$$\frac{a-b}{a} + \frac{a}{a+b} + \frac{a+b}{a+2b} + \ldots + \frac{a+n-2)b}{a+(n-1)b} = n - bS_n.$$

When $a-b$ and b are both positive, the left-hand side of the last expression will be greater than n times the geometrical average of the constituent terms, so that we shall have

$$n - bS_n > n\sqrt[n]{\frac{a-b}{a+(n-1)b}},$$

or, after transposition and division by b,

$$(2) \qquad \frac{n}{b}\left\{1 - \sqrt[n]{\frac{a-b}{a+(n-1)b}}\right\} > S_n.$$

Similarly, by adding n to both sides of (1) we deduce

$$(3) \qquad S_n > \frac{n}{b}\left\{\sqrt[n]{\frac{a+nb}{a}} = 1\right\}.$$

By combining (2) and (3) we obtain the formula

$$(a) \qquad \frac{n}{b}\left\{1 - \sqrt[n]{\frac{a-b}{a+(n-1)b}}\right\} > S_n > \frac{n}{b}\left\{\sqrt[n]{\frac{a+nb}{a}} - 1\right\}.$$

In the formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

we assume x to have a positive value and we get

$$e^x > 1 + x.$$

For values of x that are less than 3 we obtain from the formula

$$e^{-x} = 1 - x + \frac{x^2}{2!}\left(1 - \frac{x}{3}\right) + \frac{x^3}{4!}\left(1 - \frac{x}{5}\right) + \dots$$

the inequality

$$e^{-x} > 1 - x.$$

In the former of these inequalities we make x successively equal to the terms of (1), and it will result

$$e^{\frac{b}{a}} > \frac{a+b}{a}, \quad e^{\frac{b}{a+b}} > \frac{a+2b}{a+b}, \dots, e^{\frac{b}{a+(n-1)b}} > \frac{a+nb}{a+(n-1)b}.$$

All these inequalities being multiplied together, we get after reduction,

$$e^{bS_n} > \frac{a+nb}{a},$$

whence it follows

$$(4) \qquad S_n > \frac{1}{b}\{\log(a+nb) - \log a\}.$$

From the second of above inequalities, *i. e.*, $e^{-x} > 1 > x$, by putting $x = \frac{b}{a}, \frac{b}{a+b}, \frac{b}{a+2b}, \cdots$ in succession, and treating in a similar manner as in the former case, we get

$$(5) \qquad S_n < \frac{1}{b}\{\log(a + \overline{n-1}b) - \log(a-b)\}.$$

The combination of (4) and (5) gives

(b) $\frac{1}{b}\{\log(a+\overline{n-1}b)-\log(a-b)\} > S_n > \frac{1}{b}\{\log(a+nb)-\log a).$

Replacing S_n in (5) by its terms and substituting $a+b$ for a and n for $n-1$, and $1/a$ being added to both sides, the result will assume the form

(6) $$S_n < \frac{1}{a} + \frac{1}{b}\{\log(a+\overline{n-1}b)-\log a\}.$$

From (4) and (6) it follows

(c) $\frac{1}{a} + \frac{1}{b}\{\log(a+\overline{n-1}b)-\log a\} > S_n > \frac{1}{b}\{\log(a+nb)-\log a\}.$

Although the formula (a) is not convenient for practical purposes, the formulae (b) and (c) may serve for calculations, since in these the difference of the upper and lower limits, between which the value of S_n lies, may be made so as not to surpass a certain quantity for any value of n.

In fact, since b is positive as we have assumed, we have

$$\log(a+\overline{n-1}b) < \log(a+nb),$$

or

$$\frac{1}{b}\{\log(a+\overline{n-1}b)-\log a\} < \frac{1}{b}\{\log(a+nb)-\log a\},$$

so that

$$\left[\frac{1}{a}+\frac{1}{b}\{\log(a+\overline{n-1}b)-\log a\}\right] \sim \frac{1}{b}\{\log(a+nb)-\log a\}$$

is always $< \frac{1}{a}$.

Hence when the value of $\frac{1}{a}$ is very small in comparison to the value of S_n, the quantities

$$\frac{1}{a}+\frac{1}{b}\{\log(a+\overline{n-1}b)-\log a\} \text{ or } \frac{1}{b}\{\log(a+nb)-\log a\}$$

may serve for an approximate value of S_n, the deviation from the real value being less than $\frac{1}{a}$.

As the quantities $\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}, \ldots$ form a descending series of magnitudes for positive values of a and b, so making n sufficiently great, the value of $\frac{1}{a}$ may be made as small as we please in comparison to S_n. Hence in a descending harmonical progression, by

suitably choosing the number of terms, an approximate value of their sum may be obtained.

Example 1. In (c) put $a = 1$ and $b = 1$. Then, since $\log 1 = 0$, we get

$$1 + \log n > 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \log (1 + n).$$

Here the difference between the two limits is less than 1. This formula therefore enables us to obtain the value of the sum of any number of terms of our progression within the error of 1.

Example 2. If we put $a = 1000$, $b = 1$, and make $n = 10001$ in (c), we get

$$\frac{1}{1000} + \log 11000 - \log 1000 > \frac{1}{1000} + \frac{1}{1001} + \ldots (10001 \text{ terms})$$
$$> \log 11001 - \log 1000.$$

Here we have

$$\begin{aligned} \log 11000 - \log 1000 &= \log 11 \\ &= \log_{10} 11 \times \log 10 \\ &= 1 \cdot 641393 \times 2 \cdot 302585 \\ &= 2 \cdot 3979. \end{aligned}$$

Hence the sum of this progression is less than $2 \cdot 3979 + \cdot 001 = 2 \cdot 3989$, and its difference with the real value is less than $\cdot 001$. The value we have obtained is therefore correct to the second digit under the decimal point.

Now to return to the formula (a).

In the expression (2), replacing S_n by its terms, we then substitute $a + b$ for a and $n - 1$ for n, when we get

$$\frac{n-1}{b}\left\{1 - \sqrt[n-1]{\frac{a}{a+(n-1)b}}\right\} > \frac{1}{a+b} + \frac{1}{a+2b} + \ldots + \frac{1}{a+(n-1)b},$$

or

$$\frac{1}{a} + \frac{n-1}{b}\left\{1 - \sqrt[n-1]{\frac{a}{a+(n-1)b}}\right\} > S_n.$$

Hence by making a little change in the left-hand member of (a), we obtain

$$\text{(d)} \qquad \frac{1}{a} + \frac{n-1}{b}\left\{1 - \sqrt[n-1]{\frac{a}{a+(n-1)b}}\right\} > S_n > \frac{n}{b}\left\{\sqrt[n]{\frac{a+nb}{a}} - 1\right\}.$$

This formula (d) may be made to undergo a further transformation. For in the general formula

$$x = 1 + \log x + \frac{1}{2!}(\log x)^2 + \frac{1}{3!}(\log x)^3 + \ldots,$$

if we substitute

$$\sqrt[n-1]{\frac{a}{a+(n-1)b}} \text{ and } \sqrt[n]{\frac{a+nb}{a}}$$

for x, and if we denote the logarithms of these quantities by $-\frac{p}{n-1}$ and $\frac{q}{n}$ respectively, so we shall get

$$\sqrt[n-1]{\frac{a}{a+(n-1)b}} = 1 - \frac{p}{n-1} + \frac{1}{2!}\frac{p^2}{(n-1)^2} - \frac{1}{3!}\frac{p^3}{(n-1)^3} + \ldots,$$

and

$$\sqrt[n]{\frac{a}{a+(n-1)b}} = 1 + \frac{q}{n} + \frac{1}{2!}\frac{q^2}{n^2} + \frac{1}{3!}\frac{q^3}{n^3} + \ldots$$

These values being substituted, (d) becomes

$$\text{(e)} \qquad \frac{1}{a} + \frac{p}{b} - \frac{1}{2!}\frac{p^2}{(n-1)b} + \frac{1}{3!}\frac{p^3}{(n-1)^2 b} - \ldots$$

$$> S_n > \frac{q}{b} + \frac{1}{2!}\frac{q^2}{nb} + \frac{1}{3!}\frac{q^3}{n^2 b} + \ldots,$$

where we write

$$p = \log\{a + (n-1)b\} - \log a,$$

and

$$q = \log\{a + nb\} - \log a.$$

Again in (2) make the substitution $a = a + b$, when it becomes

$$\frac{n}{b}\left\{1 - \sqrt[n]{\frac{a}{a+nb}}\right\} > \frac{1}{a+b} + \frac{1}{a+2b} + \ldots + \frac{1}{a+nb},$$

whence we deduce

$$\text{(f)} \quad \frac{1}{a} + \frac{n}{b}\left\{1 - \sqrt[n]{\frac{a}{a+nb}}\right\} - \frac{1}{a+nb} > S_n > \frac{n}{b}\left\{\sqrt[n]{\frac{a+nb}{a}} - 1\right\}.$$

The expression

$$\sqrt[n]{\frac{a}{a+nb}} = 1 - \frac{q}{n} + \frac{1}{2!}\frac{q^2}{n^2} - \frac{1}{3!}\frac{q^3}{n^3} + \ldots$$

is obvious from a formula already employed. When this is substituted in (f), we obtain

$$\text{(g)} \qquad \frac{1}{a} - \frac{1}{a+nb} + \frac{q}{b} - \frac{1}{2!}\frac{q^2}{nb} + \frac{1}{3!}\frac{q^3}{n^2 b} - \ldots$$

$$> S_n > \frac{q}{b} + \frac{1}{2!}\frac{q^2}{nb} + \frac{1}{3!}\frac{q^3}{n^2 b} + \ldots$$

For the purpose of calculating a value approximate to S_n, the formula (g) will be evidently more convenient than (d)

Example 3. Here we have to calculate the second example by an application of the formula (g). Necessary substitutions being made, we have

$$\frac{1}{1000} - \frac{1}{11000} + q - \frac{1}{2!}\frac{q^2}{1000} + \frac{1}{3!}\frac{q^3}{1000^2} - \dots$$
$$> \frac{1}{1000} + \frac{1}{1001} + \dots + \frac{1}{10999}$$
$$> q + \frac{1}{2!}\frac{q^2}{1000} + \frac{1}{3!}\frac{q^3}{1000^3} + \dots,$$

where

$$q = \log 11000 - \log 1000$$
$$= \log 11 = 2\cdot 397895.$$

Add $\frac{1}{11000}$ to each member of above inequality, and we have

$$2\cdot 39861 > \frac{1}{1000} + \frac{1}{1001} + \dots + \frac{1}{11000} > 2\cdot 39827.$$

The average of the two limits being taken, we find the sum required at the value of $2\cdot 3984$. This approximation is correct to three decimal places.

II.

Some years since U. Fujimaki had made public his researches on the limits of the sum of a harmonic progression, there appeared in the same *Journal* an essay contributed by T. Kariya concerning on the same subject. His paper is entitled *On the sum of a harmonical progression* and we find it in the January number of the *Journal* in 1904.

T. Kariya begins thus in his paper:

Chrystal says in his *Text-Book of Algebra, vol. I, p. 278* that the formula that expresses the sum of n terms in a harmonical progression has various forms. From this it will be evident that there are several expressions already established for such a sum. Here I give a result of my study, which is however no more than a rewriting of the series.

Writing

$$S = \frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \dots + \frac{1}{a+(n-1)b},$$

we have

$$S = \int_0^1 (x^{a-1} + x^{a+b-1} + x^{a+2b-1} + \dots + x^{a+(n-1)b-1})\,dx.$$

But as we know

$$x^{a-1} + x^{a+b-1} + \dots + x^{a+(n-1)b-1} = x^{a-1}\,\frac{1-x^{nb}}{1-x^{b}},$$

so that

$$S = \int_0^1 x^{a-1} \cdot \frac{1-x^{nb}}{1-x^b}\, dx$$
$$= \int_0^1 \frac{x^{a-1}}{1-x^b}\, dx - \int_0^1 \frac{x^{a+nb-1}}{1-x^b}\, dx.$$

The substitution $x^b = y$ transforms this into

$$S = -\frac{1}{b}\int_0^1 \frac{1-y^{\frac{a}{b}-1}}{1-y}\, dy + \frac{1}{b}\int_0^1 \frac{1-y^{\frac{a+nb}{b}-1}}{1-y}\, dy,$$

which, expressed by aid of the gamma function, becomes

$$S = \frac{1}{b}\left\{\frac{d\log\Gamma(x)}{dx}\right\}_{x=\frac{a+nb}{b}} - \frac{1}{b}\left\{\frac{d\log\Gamma(x)}{dx}\right\}_{x=\frac{a}{b}},$$

or

$$S = \frac{1}{b}\int_{\frac{a}{b}}^{\frac{a+nb}{b}} \frac{d^2\log\Gamma(x)}{dx^2}\, dx.$$

Note. Since

$$\frac{d\log\Gamma(x)}{dx} = -C + \left(1-\frac{1}{x}\right) + \left(\frac{1}{2}-\frac{1}{x+1}\right) + \left(\frac{1}{3}-\frac{1}{x+2}\right) + \ldots,$$

$$\therefore\ Sb = -C + \left(1-\frac{b}{a+nb}\right) + \left(\frac{1}{2}-\frac{b}{a+(n-1)b}\right) + \left(\frac{1}{3}-\frac{b}{a+(n-2)b}\right) + \ldots$$
$$+ C - \left(1-\frac{b}{a}\right) - \left(\frac{1}{2}-\frac{b}{a+b}\right) - \left(\frac{1}{3}-\frac{b}{a+2b}\right) - \ldots$$
$$= \frac{b}{a} + \frac{b}{a+b} + \frac{b}{a+2b} + \ldots + \frac{b}{a+(n-1)b},$$

that is, it gets on the original expression we have started with.

It is therefore doubtful whether the sum of a number of terms that form a harmonic progression can be found in the sense as it will be said about an arithmetic or geometric progression.

III.

U. Fujimaki, after the publication of his first paper, had continued in his study of the summation of the harmonical progression, when he achieved on the basis of a formula of Euler a method of giving an expression in the form of an infinite series for the sum under consideration. And now T. Kariya's preceding paper being published he soon afterward brought out the result of his subsequent studies.[1])

1) Journal of Physics School, Vol. 13, pp. 208—213, May, 1904.

The said formula of Euler is this:

$$\Sigma\Phi(x) = C + \frac{1}{h}\int\Phi(x)\,dx - \frac{1}{2}\Phi(x) + B_1\Phi'(x)\frac{h}{2!} - B_3\Phi'''(x)\frac{h^3}{4!} + B_5\Phi^{(5)}(x)\frac{h^5}{6!} - \dots + (-1)^{n+1}B_{2n-1}\Phi^{(2n-1)}(x)\frac{h^{2n-1}}{(2n)!} + \dots,$$

where $x = a + nh$, *n being a positive integer, and*

$$\Sigma\Phi(x) = \Phi(a) + \Phi(a+h) + \Phi(a+2h) + \dots + \Phi(x-h),$$

and

$$C = -\frac{1}{h}\int\Phi(a)\,da + \frac{1}{2}\Phi(a) - B_1\Phi'(a)\frac{h}{2!} + B_3\Phi'''(a)\frac{h^3}{4!} - B_5\Phi^{(5)}(a)\frac{h^5}{6!} + \dots,$$

where $B_1, B_3, B_5, \dots$ *are Bernouilli's numbers.*

If we put $\Phi(x) = \frac{1}{x}$, we have

$$\Sigma\Phi(x) = \frac{1}{a} + \frac{1}{a+h} + \frac{1}{a+2h} + \dots + \frac{1}{x-h},$$

$$\int\Phi(x)\,dx = \int\frac{dx}{x} = \log x,$$

and

$$\Phi^{(r)}(x) = \frac{d^r(x^{-1})}{dx^r} = \frac{(-1)^r r!}{x^{r+1}},$$

so that the above formula reduces to

$$\frac{1}{a} + \frac{1}{a+h} + \frac{1}{a+2h} + \dots + \frac{1}{x-h} = C + \frac{1}{h}\log x - \frac{1}{2}\frac{1}{x} + B_1\frac{-1}{x^2}\frac{h}{2!} - B_3\frac{-3!}{x^4}\frac{h^3}{4!} + \dots + (-1)^{n+1}B_{2n-1}\frac{-(2n-1)!}{x^{2n}}\frac{h^{2n-1}}{(2n)!} + \dots,$$

that is,

$$(1)\qquad \frac{1}{a} + \frac{1}{a+h} + \frac{1}{a+2h} + \dots + \frac{1}{a+(n-1)h} = C + \frac{1}{h}\log(a+nh) - \frac{1}{2}\frac{1}{a+nh} - \frac{B_1}{2}\frac{h}{(a+nh)^2} + \frac{B_3}{4}\frac{h^3}{(a+nh)^4} - \frac{B_5}{6}\frac{h^5}{(a+nh)^6} + \dots + (-1)^n\frac{B_{2n-1}}{2n}\frac{h^{2n-1}}{(a+nh)^{2n}} + \dots,$$

where

$$(1')\qquad C = -\frac{1}{h}\log a + \frac{1}{2}\frac{1}{a} + \frac{B_1}{2}\frac{h}{a^2} - \frac{B_3}{4}\frac{h^3}{a^4} + \frac{B_5}{6}\frac{h^5}{a^6} - \dots$$

By putting $h = 1$ in this formula we obtain

$$(2) \qquad \frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \dots + \frac{1}{a+n-1}$$
$$= C + \log(a+n) - \frac{1}{2}\frac{1}{a+n} - \frac{B_1}{2}\frac{1}{(a+n)^2}$$
$$+ \frac{B_3}{4}\frac{1}{(a+n)^4} - \frac{B_5}{6}\frac{1}{(a+n)^6} + \dots,$$

where

$$(2') \qquad C = -\log a + \frac{1}{2}\frac{1}{a} + \frac{B_1}{2}\frac{1}{a^2} - \frac{B_3}{4}\frac{1}{a^4} + \frac{B_5}{6}\frac{1}{a^6} - \dots$$

We have also

$$\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \dots + \frac{1}{a+(n-1)d}$$
$$= \frac{1}{d}\left\{\frac{1}{\frac{a}{d}} + \frac{1}{\frac{a}{d}+1} + \frac{1}{\frac{a}{d}+2} + \dots + \frac{1}{\frac{a}{d}+(n-1)}\right\},$$

or the quantity within the crooked brackets being compared with (2) and $\frac{a}{d}$ being denoted by b,

$$(3) \qquad = \frac{1}{d}\left\{C + \log(b+n) - \frac{1}{2}\frac{1}{b+n} - \frac{B_1}{2}\frac{1}{(b+n)^2}\right.$$
$$\left. + \frac{B_3}{4}\frac{1}{(b+n)^4} - \frac{B_5}{6}\frac{1}{(b+n)^6} + \dots\right\},$$

where

$$(3') \qquad C = -\log b + \frac{1}{2}\frac{1}{b} + \frac{B_1}{2}\frac{1}{b^2} - \frac{B_3}{4}\frac{1}{b^4} + \frac{B_5}{6}\frac{1}{b^6} - \dots$$

The formulae (1), (2) and (3) express the sum of n terms of a harmonic progression in infinite series. These series are all convergent and their degrees of convergency increase with the values of n. The values for C are convergent when $\frac{h}{a}$, a, b are not less than unity; but there may happen cases where they are still convergent for values of $\frac{h}{a}$, a, or b, that are less than unity. To procure a rapid convergency, however, it is necessary to take large values for these quantities.

If we put $a = 1$ in (2), we have

$$(4) \qquad 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$
$$= C + \log(n+1) - \frac{1}{2}\frac{1}{n+1} - \frac{B_1}{2}\frac{1}{(n+1)^2}$$
$$+ \frac{B_3}{4}\frac{1}{(n+1)^4} - \frac{B_5}{6}\frac{1}{(n+1)^6} + \dots,$$

where C assumes a value that is usually called by the name of Euler's constant, namely, $\cdot 5772156649\dots$

When we have to calculate the right-hand member of (3), if the value of b is not sufficiently great and so the right-hand side of (3′) converges very slowly, we can better strike the following way.

We have from (2)

$$\frac{1}{a}+\frac{1}{a+1}+\frac{1}{a+2}+\dots+\frac{1}{a+(r-1)}$$
$$=C+\log(a+r)-\frac{1}{2}\frac{1}{a+r}-\frac{B_1}{2}\frac{1}{(a+r)^2}$$
$$+\frac{B_3}{4}\frac{1}{(a+n)^4}-\frac{B_5}{6}\frac{1}{(a+r)^6}+\dots$$

Subtracting this from (2), C having the same value in both and r being less than n, we have

$$\frac{1}{a+r}+\frac{1}{a+r+1}+\frac{1}{a+r+2}+\dots+\frac{1}{a+(n-1)}$$
$$=\log(a+n)-\frac{1}{2}\frac{1}{a+n}-\frac{B_1}{2}\frac{1}{(a+n)^2}+\frac{B_3}{4}\frac{1}{(a+n)^4}-\frac{B_5}{6}\frac{1}{(a+n)^6}+\dots$$
$$-\log(a+r)+\frac{1}{2}\frac{1}{a+r}+\frac{B_1}{2}\frac{1}{(a+r)^2}-\frac{B_3}{4}\frac{1}{(a+r)^4}+\frac{B_5}{6}\frac{1}{(a+r)^6}-\dots$$
$$=C+\log(a+n)-\frac{1}{2}\frac{1}{a+n}-\frac{B_1}{2}\frac{1}{(a+n)^2}+\frac{B_3}{4}\frac{1}{(a+n)^4}-\frac{B_5}{6}\frac{1}{(a+n)^6}+\dots$$

Now a being replaced by $b\left(=\frac{a}{d}\right)$ we have

$$\frac{1}{b+r}+\frac{1}{b+r+1}+\dots+\frac{1}{b+n-1}$$
$$=C+\log(b+n)-\frac{1}{2}\frac{1}{b+n}-\frac{B_1}{2}\frac{1}{(b+n)^2}+\frac{B_3}{4}\frac{1}{(b+n)^4}-\frac{B_5}{6}\frac{1}{(b+n)^6}+\dots,$$

where

$$C=-\log(b+r)+\frac{1}{2}\frac{1}{b+r}+\frac{B_1}{2}\frac{1}{(b+r)^2}-\frac{B_3}{4}\frac{1}{(b+r)^4}+\frac{B_5}{6}\frac{1}{(b+r)^6}-\dots$$

We have then

$$(5)\quad \frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\dots+\frac{1}{a+(n-1)d}$$
$$=\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\dots+\frac{1}{a+(r-1)d}$$
$$+\frac{1}{d}\left\{\frac{1}{b+r}+\frac{1}{b+r+1}+\dots+\frac{1}{b+n-1}\right\}$$
$$=\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\dots+\frac{1}{a+(r-1)d}$$
$$+\frac{1}{d}\left\{C+\log(b+n)-\frac{1}{2}\frac{1}{b+n}-\frac{B_1}{2}\frac{1}{(b+n)^2}+\frac{B_3}{4}\frac{1}{(b+n^4}\right.$$
$$\left.-\frac{B_5}{6}\frac{1}{(b+n)^6}+\dots\right\},$$

where

$$(5')\quad C=-\log(b+r)+\frac{1}{2}\frac{1}{b+r}+\frac{B_1}{2}\frac{1}{(b+r)^2}-\frac{B_8}{4}\frac{1}{(b+r)^4}+\frac{B_5}{6}\frac{1}{(b+r)^6}-\dots$$

The value of C in $(5')$ will be seen considerably rapid in its degree of convergency when compared to that in $(3')$. Consequently we have more advantages in the employment of (5) and $(5')$ over (3), when the value of b is not great enough. In this case we must however directly calculate the sum of the first r terms, the number r being to be taken at will. Of course the greater the number r the more rapid the convergency of C.

Example 1. To calculate

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}.$$

We have from (4)

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}=C+\log 11-\frac{1}{2}\frac{1}{11}+\frac{B_1}{2}\frac{1}{11^2}$$
$$+\frac{B_3}{4}\frac{1}{11^4}-\frac{B_5}{6}\frac{1}{11^6}+\ldots$$
$$=C+\frac{\log_{10}11}{\log_{10}e}-\frac{1}{2}\frac{1}{11}-\frac{1}{12}\frac{1}{11^2}+\frac{1}{120}\frac{1}{11^4}-\frac{1}{252}\frac{1}{11^6}+\ldots$$
$$=\cdot 5772+2\cdot 3979-\cdot 0455-\cdot 0007+\ldots$$
$$=2\cdot 9289\ldots$$

Example 2. To find a hundred terms of

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\ldots$$

Here we employ (5) and $(5')$. Leaving the first seven terms for a direct calculation, we have

$$\frac{1}{23}+\frac{1}{26}+\frac{1}{29}+\ldots+\frac{1}{299}$$
$$=\frac{1}{3}\left\{C+\log\left(\frac{2}{3}+100\right)-\frac{1}{2}\frac{1}{\frac{2}{3}+100}-\frac{1}{12}\frac{1}{\left(\frac{2}{3}+100\right)^2}+\ldots\right\},$$

where

$$C=-\log\left(\frac{2}{3}+7\right)+\frac{1}{2}\frac{1}{\frac{2}{3}+7}+\frac{1}{12}\frac{1}{\left(\frac{2}{3}+7\right)^2}-\frac{1}{121}\frac{1}{\left(\frac{2}{3}+7\right)^4}+\ldots$$
$$=-2\cdot 0369+\cdot 0652+\cdot 0014-\ldots$$
$$=-1\cdot 9703.$$

Hence

$$\frac{1}{23}+\frac{1}{26}+\ldots+\frac{1}{299}=\frac{1}{3}(-1\cdot 9703+4\cdot 6119-\cdot 0050-\ldots)$$
$$=\cdot 8789\ldots$$

The sum of the first 7 terms is

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\ldots+\frac{1}{20}=1\cdot 0962.$$

The required sum is therefore

$$1\cdot 0962+\cdot 8789=1\cdot 9751,$$

where the last digit is not necessarily exact.

T. HAYASHI, ON A CONVERGENCY-TEST OF INFINITE SERIES WITH POSITIVE TERMS.[1])

Cauchy proved that the infinite series with positive terms, Σa_n, is convergent or divergent according as $\lim \sqrt[n]{a_n} <> 1$.

The test we are going here to give is merely a simple transformation of this test of Cauchy's.

Let $f(n)$ be a function such that $\lim f(n)=\infty$, and let us write

$$f(n)\left\{1-a_n^{\frac{1}{nf(n)}}\right\}=r_n,$$

or

$$\sqrt[n]{a_n}=\left\{1-\frac{r_n}{f(n)}\right\}^{f(n)}.$$

We have then

$$\lim \sqrt[n]{a_n}=\lim e^{-r_n}=e^{-\lim r_n}.$$

The series Σa_n *is therefore convergent or divergent according as* $\lim r_n ><0$, *that is, according as*

$$\lim f(n)\left\{1-a_n^{\frac{1}{nf(n)}}\right\}><0.$$

The advantage of this convergency-test lies in the arbitrariness of the function $f(n)$.

The following series suit themselves to a test by our criterion:

$$\sum\left\{\frac{(2n-1)^2}{2n(2n+1)}\right\}^{n^2},\quad \sum\left\{\frac{\log(kn-1)}{\log n}\right\}^{n\log n},$$

$$\sum\left\{\frac{\log n}{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}\right\}^{n\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)},$$

1) Journ. of Phys. Sch., Vol. 7, pp. 161—162, April, 1903.

$$\sum\left\{1+\frac{\Phi(\sqrt{n})}{\sqrt{n}\,\Psi(\sqrt{n})}\right\}^{n\sqrt{n}},$$

where Φ and Ψ are rational functions of the same degree.

T. HAYASHI, ON THE THEOREMS ON THE MEANS.

The following is the result of two papers published by T. Hayashi in the *Journal of the Physics School in Tokyo*, Vol. 7, pp. 12—14, December, 1897 and Vol. 13, pp. 46 – 50, January, 1904.

1. From the theorem, that *the arithmetic mean of a number of positive quantities is never less than their geometric mean*, we deduce another, which is new.

If the aggregate of the geometric means of every q out of n positive quantities $a_1, a_2, \ldots, a_n$ will be denoted by $\Sigma_q\sqrt[q]{a_1a_2\ldots a_q}$ and if ${}_nC_q$ stand for the number of combinations of taking q from n, then we shall have

$$\frac{\Sigma_q\sqrt[q]{a_1a_2\ldots a_q}}{{}_nC_q} \not< \frac{\Sigma_{q+\lambda}\sqrt[q+\lambda]{a_1a_2\ldots a_{q+\lambda}}}{{}_nC_{q+\lambda}}$$

$$(q,\ q+\lambda < n).$$

We have only to prove the proposition for the case $\lambda = 1$. Consequently it will suffice to prove

$$\Sigma_{q+1}\sqrt[q+1]{a_1a_2\ldots a_{q+1}} \not> \frac{n-q}{q+1}\Sigma_q\sqrt[q]{a_1a_2\ldots a_q}.$$

Now if we select q out of $q+1$ quantities and form their geometric mean, there will be $q+1$ of such means. We denote the product and the sum of these geometric means by $\Pi\sqrt[q]{a_1a_2\ldots a_q}$ and $\Sigma\sqrt[q]{a_1a_2\ldots a_q}$, respectively. Then we have

$$a_1a_2\ldots a_{q+1} = \Pi\sqrt[q]{a_1a_2\ldots a_q}.$$

Hence

$$\sqrt[q+1]{a_1a_2\ldots a_{q+1}} \not> \frac{\Sigma\sqrt[q]{a_1a_2\ldots a_q}}{q+1}.$$

Hence $\Sigma_{q+1}\sqrt[q+1]{a_1a_2\ldots a_{q+1}}$ is not greater than the sum of $(q+1)_nC_{q+1}$ terms that have the forms like $\dfrac{\sqrt[q]{a_1a_2\ldots a_q}}{q+1}$. But the n given quantities will give ${}_nC_q$ of such terms and it is

$$(q+1)_nC_{q+1} = (n-q)_nC_q,$$

so that

$$\Sigma_{q+1}\sqrt[q+1]{a_1a_2\ldots a_{q+1}} \not> \frac{n-q}{q+1}\Sigma_q\sqrt[q]{a_1a_2\ldots a_q}.$$

2. The above theorem may be enunciated as follows:

If G_n^r denote the sum of the geometric means of every r out of n positive quantities $a_1, a_2, \ldots, a_n$, and if C_n^r stand for the combinations of r out of n, and if we put $G_n^r \div C_n^r = K_r$, then

$$K_1 \prec K_2 \prec K_3 \ldots \prec K_n.$$

In correspondence to this theorem we have another one:

We form the arithmetic mean of r taken from n different positive quantities and take A_n^r for the product of all such quantities, and put

$$\left(A_n^r\right)^{\frac{1}{C_n^r}} = L_r,$$

then we have

$$L_1 \prec L_2 \prec L_3 \ldots \prec L_n.$$

This theorem will be proved subsequently, but meanwhile let us look a little around us.

A. Durand gives a proof to the following theorem[1]):

If S_n^r denote the sum of the products of r factors taken out of n different positive quantities, and if we put

$$\sqrt[r]{\frac{S_n^r}{C_n^r}} = M_r,$$

then

$$M_1 > M_2 > M_3 \ldots > M_n.$$

Corresponding to this we have the theorem:

We form the sum of r taken from n different positive quantities, and represent the product of all such possible sums by P_n^r and put

$$\frac{\left(P_n^r\right)^{\frac{1}{C_n^r}}}{r} = N_r,$$

then

$$N_1 < N_2 < N_3 \ldots < N_n.$$

When we compare this theorem to that we have given above without proof, it will be seen both express the same fact, that is, it may be considered as the same theorem that corresponds at the same time to two different theorems. It will therefore suffice to prove in one of its forms. We select the latter.

1) Bulletin des Sciences Mathematiques. (2) T. XXVI. *Sur un théorème relatif à des moyennes.*

We first assume the relation

$$\frac{\left(P_n^r\right)^{\frac{1}{C_n^r}}}{r} < \frac{\left(P_n^{r+1}\right)^{\frac{1}{C_n^{r+1}}}}{r+1} \qquad (r \leq n-1)$$

to hold good.

By replacing successively one of $a_1, a_2, \ldots, a_n$ by a_{n+1}, and multiplying together the corresponding members of the n inequalities thus obtained and of the above one, we obtain

$$\frac{\left(\Pi P_n^r\right)^{\frac{1}{C_n^r}}}{r^{n+1}} < \frac{\left(\Pi P_n^{r+1}\right)^{\frac{1}{C_n^{r+1}}}}{(r+1)^{n+1}}.$$

The quantity ΠP_n^r evidently contains $(n+1)\,C_n^r$ factors, and also contains every factor of the C_{n+1}^r contained in P_{n+1}^r, so that

$$\Pi P_n^r = \left(P_{n+1}^r\right)^{\frac{(n+1)\,C_n^r}{C_{n+1}^r}}.$$

Similarly we have

$$\Pi P_n^{r+1} = \left(P_{n+1}^{r+1}\right)^{\frac{(n+1)\,C_n^{r+1}}{C_{n+1}^{r+1}}}.$$

$$\therefore \frac{\left(P_{n+1}^r\right)^{\frac{n+1}{C_{n+1}^r}}}{r^{n+1}} < \frac{\left(P_{n+1}^{r+1}\right)^{\frac{n+1}{C_{n+1}^{r+1}}}}{(r+1)^{n+1}}.$$

$$\therefore \frac{\left(P_{n+1}^r\right)^{\frac{1}{C_{n+1}^r}}}{r} < \frac{\left(P_{n+1}^{r+1}\right)^{\frac{1}{C_{n+1}^{r+1}}}}{r+1}$$

$$(r < n-1).$$

The correctness of this formula for $r = n$ can be directly proved. Therefore the method of mathematical induction applies.

3. *The following way of a direct demonstration is due to K. Katō.*

If Q_n^r denote the sum of the sums of r taken from n quantities, it is evidently

$$\left(P_n^r\right)^{\frac{1}{C_n^r}} < \frac{Q_n^r}{C_n^r}.$$

It is also evident that Q_n^r is a multiple of $\sum_n a$ and that

$$Q_n^r = \frac{r}{n} C_n^r \sum_n a.$$

If we put $r = n - 1$ or $n = r + 1$, we get therefore

$$\left(P_{r+1}^r\right)^{\frac{1}{C_{r+1}^r}} < \frac{r}{r+1} \sum_{r+1} a.$$

If we next denote the product of the sums constructed by taking $r + 1$ from the quantities $a_1, a_2, \ldots$ by $\Pi(a_1 + a_2 + \ldots + a_{r+1})$, we have from the above inequality

$$\Pi\left(P_{r+1}^r\right)^{\frac{1}{C_{r+1}^r}} < \left(\frac{r}{r+1}\right)^{C_n^{r+1}} \Pi(a_1 + a_2 + \ldots + a_{r+1}),$$

where $\Pi(a_1 + a_2 + \ldots + a_{r+1})$ of course represents P_n^{r+1}.

Hence it follows

$$\Pi\left(P_{r+1}^r\right)^{\frac{1}{C_{r+1}^r}} < \left(\frac{r}{r+1}\right)^{C_n^{r+1}} P_n^{r+1}.$$

It is evident that $\Pi(P_{r+1}^r)$ is a symmetrical expression of $a_1, a_2, \ldots, a_n$ and equal to an integral power of P_n^r. It is also evident that the same quantity contains $C_{r+1}^r \cdot C_n^{r+1}$ factors.

But P_n^r contains C_n^r factors.

We have therefore

$$\left\{\Pi\left(P_{r+1}^r\right)\right\}^{C_n^r} = \left(P_n^r\right)^{C_{r+1}^r C_n^{r+1}}.$$

$$\therefore \left\{\Pi\left(P_{r+1}^r\right)\right\}^{\frac{1}{C_{r+1}^r}} = \left(P_n^r\right)^{\frac{C_n^{r+1}}{C_n^r}}.$$

$$\therefore \left(P_n^r\right)^{\frac{C_n^{r+1}}{C_n^r}} < \left(\frac{r}{r+1}\right)^{C_n^{r+1}} \times P_n^{r+1}.$$

And thence finally

$$\frac{\left(P_n^r\right)^{\frac{1}{C_n^r}}}{r} < \frac{\left(P_n^{r+1}\right)^{\frac{1}{C_n^{r+1}}}}{r+1},$$

as we had to prove.

T. YOSHIYE, THE SOLUTION OF AN EQUATION CONSIDERED AS A MAXIMUM AND MINIMUM PROBLEM.[1)]

1. There are problems that can be deduced to the treatment of maxima and minima. The solution of an equation may be brought under such a domain. Here we propose to say something about the matter. Although our consideration is applicable in the general case, but for the sake of simplicity, we try to treat only for the case of the quadratic equation $ax^2 + 2bx + c = 0$.

In a rectangular system of coordinates we take the curve

$$y = ax^2 + 2bx + c,$$

and denote by A and B the points where it cuts the axis of x.

At a point P on the curve near B we draw the tangent p, that cuts the axes of x and y in Q and M respectively. Let the tangent at B intersect with the y-axis in M_1.

When the point P is conceived to move along the curve towards B, the point Q approaches along the x-axis towards B and after coming into coincidence with it, then retreats the same path. Therefore $OQ = \alpha$ has a maximum or a minimum value at B. The same will be said also for the point A. (This consideration does not apply when A or B is a point of inflexion or a cusp.)

OM being denoted by β, α is a maximum or minimum, when β becomes $= OM_1$.

If we write
$$-\frac{1}{\alpha} = u, \quad -\frac{1}{\beta} = v,$$

then (u, v) represent the line-coordinates of p. If we express the condition of contact of this straight line (u, v) with the curve, we get its equation referred to the line-coordinates. The curve is the envelope of the straight line that satisfies this equation.

The original equation of the curve being $y = ax^2 + 2bx + c$, the equation in the line-coordinates is

$$\begin{vmatrix} a & 0 & b & u \\ 0 & 0 & -\frac{1}{2} & v \\ b & -\frac{1}{2} & c & 1 \\ u & v & 1 & 0 \end{vmatrix} = 0,$$

or

$$f(u, v) = \frac{u^2}{4} + buv + (b^2 - ac)v^2 - av = 0.$$

1) Journal of Tokyo Physics School, Vol. 15, pp. 464—468, November, 1906.

Since α the function of β is maximum or minimum at A and B, u the function of v is also maximum or minimum at these points, so that we shall have $\frac{du}{dv} = 0$. Hence differentiating $f(u, v) = 0$, we get

$$\frac{\partial f}{\partial v} = bu + 2(b^2 - ac)v - a = 0,$$

whence follows

$$v = \frac{a - bu}{2(b^2 - ac)}.$$

This value being substituted in $f(u, v) = 0$, we have

$$u^2(b^2 - ac) = (a - bu)^2,$$

or

$$u = \frac{a}{b \pm \sqrt{b^2 - ac}}$$

But since $u = -\frac{1}{\alpha}$, we have as the maximum or minimum values of α, that is, as the two roots of $ax^2 + 2bx + c = 0$, the values

$$\frac{-b \pm \sqrt{b^2 - ac}}{a}.$$

2. To a maximum or minimum value of y there will correspond in general a tangent to the curve that is parallel to the axis of x, or $u = 0$ for such a tangent. And conversely, when $u = 0$, y has in general a maximum or a minimum value. Therefore, if we write $u = 0$ in the equation in the line-coordinates, and calculate the value of $-\frac{1}{v}$ that arises therefrom, we get the maximum or minimum value of y.

In the case before us, we write $u = 0$ in

$$\frac{u^2}{4} + buv + (b^2 - ac)v^2 - av = 0$$

and we get

$$\{(b^2 - ac)v - a\}v = 0,$$

whence

$$v = 0 \quad \text{or} \quad v = \frac{a}{b^2 - ac},$$

that is, the maximum and minimum values of y are ∞ and $\frac{ac - b^2}{a}$.

SUIHOKU, SERIES THAT GIVE THE VALUES OF $\frac{1}{\pi}$.[1]

By the binomial theorem we have

$$\text{(A)} \quad \sqrt{1 - \sin^2 \Phi} = 1 - \frac{1}{2}\sin^2 \Phi - \frac{1}{2\cdot 4}\sin^4 \Phi - \frac{1\cdot 3}{2\cdot 4\cdot 6}\sin^6 \Phi - \dots$$

1) The Journal of the Society of Mathematics in Tokyo, Vol. 5, pp. 283—284, 1892. Given anonymous under the nom-de-plume, Suihoku-Koji, probably S. Shitō.

But since

$$\int_0^{\frac{\pi}{2}} \sin^{2m} \Phi d\Phi = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\cdot\frac{\pi}{2},$$

the formula (A) being integrated between the limits 0 and $\frac{\pi}{2}$ for each term, there arises the identity

$$1 = \frac{\pi}{2}\left\{1 - \frac{1}{2^2} - \frac{1^2\cdot 3}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} - \cdots\right\},$$

or what is the same thing,

$$\text{(B)} \qquad \frac{1}{\pi} = \frac{1}{2}\left\{1 - \frac{1}{2^2} - \frac{1^2\cdot 3}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2} - \cdots\right\}.$$

Since the series in (A) is convergent, it may be believed at once that the series in (B) is also convergent.

But the convergency of (B) may be directly proved; for the series within the brackets being rewritten assumes the form

$$1 + \frac{\left(-\frac{1}{2}\right)\frac{1}{2}}{1\times 1} + \frac{\left(-\frac{1}{2}\right)\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2\times 1\cdot 2} + \cdots,$$

which is a hypergeometric series and evidently convergent.

The formula (B), therefore, represents the true value of $1/\pi$.

Taking in place of $\sqrt{1-\sin^2\Phi}$ the function $\sqrt{1-\sin^2\Phi}\cos^{2n}\Phi$, we can proceed in the same manner as in the above.

Thus we integrate each term in the expansion of the said function by applying to the formula

$$\int_0^{\frac{\pi}{2}} \sin^{2m}\Phi\cos^{2n}\Phi d\Phi = \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2m+2n)}\cdot\frac{\pi}{2},$$

and we get

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\Phi}\cos^{2n}\Phi d\Phi$$

$$= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{\pi}{2}\cdot\left\{1 - \frac{1}{2}\,\frac{1}{2n+2} - \frac{1}{2\cdot 4}\,\frac{1\cdot 3}{(2n+2)(2n+4)}\right.$$
$$\left. - \frac{1\cdot 3}{2\cdot 4\cdot 6}\,\frac{1\cdot 3\cdot 5}{(2n+2)(2n+4)(2n+6)} - \cdots\right\}.$$

But

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} \Phi d\Phi = \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots (2n+1)}.$$

Hence

$$(C) \quad \frac{1}{\pi} = \left(\frac{1\cdot 3\cdot 5\cdots (2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right)^2 \frac{2n+1}{2} \times \left\{1 - \frac{1}{2}\,\frac{1}{2n+2} - \frac{1}{2\cdot 4}\,\frac{1\cdot 3}{(2n+2)(2n+4)} - \frac{1\cdot 3}{2\cdot 4\cdot 6}\,\frac{1\cdot 3\cdot 5}{(2n+2)(2n+4)(2n+6)} - \cdots\right\}.$$

About the convergency of this series we can say the same thing as we have done above.

T. FUJII, ON THE INTEGRAL

$$u = \int_0^x \frac{dx}{\sqrt{1-x^2}}.\text{[1]}$$

The integral

$$u = \int_0^x \frac{dx}{\sqrt{1-x^2}}$$

is usually at once put equal to the function arc sin x, and no other proceedings are sought for, as the author maintains. But a change of the variable will easily enable us to transform the integral, which is likewise immediately integrable. Thus substituting for x the function

$$x = \frac{Az^2+B}{z},$$

where A and B are constants to be determined, we have after reduction

$$u = \int_{\sqrt{-\frac{B}{A}}}^{z} \frac{\frac{Az^2-B}{z}}{\sqrt{z^2-(Az^2+B)^2}}\,dz.$$

Here we have to choose for A and B such values that the quantity under the radical sign should become a perfect square; a condition that is to be determined from the relation

$$A^2B^2 = \frac{1}{4}(2AB-1)^2, \text{ or } AB = \frac{1}{4}.$$

1) The Journal of Phys. Sch. in Tokyo, Vol. 4, pp. 337—338, 1895.

Thus the expression under the symbol of integration reduces to $\frac{dz}{iz}$. Hence we have on integrating

$$iu = \log z - \log \sqrt{-\frac{B}{A}} = \log z \sqrt{-\frac{A}{B}}$$
$$= \log 2iAz = \log \frac{iz}{2B}.$$

But $Az^2 - xz + B = 0$, as we have assumed it. Hence

$$z = \frac{x \pm \sqrt{x^2 - 1}}{2A},$$

where the upper or lower sign should be suitably selected. To that purpose we take the negative sign. Therefore

$$iu = \log \left\{ \sqrt{-\frac{A}{B}} \frac{x - \sqrt{x^2-1}}{2A} \right\}$$
$$= \log i \left(x - i\sqrt{1 - x^2}\right) = \log \left(\sqrt{1 - x^2} + ix\right).$$

TANAKA, A REMARK ON WILLIAMSON'S INTEGRAL CALCULUS.[1]

We find in B. Williamson's *Treatise on the Integral Calculus*[2]), p. 139, the formula

$$(\alpha) \qquad \int_0^\infty \frac{dx}{1 - x^2} = 0,$$

which is nothing but a gross mistake.

Here let us see how Williamson concerns with his reasoning on this matter.

By the substitution $x = \frac{1}{z}$, we get

$$\int_1^\infty \frac{dx}{1 - x^2} = \int_1^0 \frac{dz}{1 - z^2} = -\int_0^1 \frac{dz}{1 - z^2} = -\int_0^1 \frac{dx}{1 - x^2},$$

and therefore

$$\int_0^\infty \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^\infty \frac{dx}{1 - x^2} = 0.$$

Williamson makes also the use of his formula thus obtained to deduce another, namely,

$$(\beta) \qquad \pi \cot a\pi = \int_0^\infty \frac{x^{a-1}}{1 - x} dx.$$

1) The Journal of Phys. Sch. in Tokyo, Vol. 8, pp. 12—14. December 1898.

2) This work was widely studied in Japan.

The formula (α) is already an error. How can then (β) hold good, when it lays its basis on another that is false?

The true value of (β) is said to be found in the expression

$$\pi \cot a\pi = \int_0^\infty \frac{x^{-\frac{1}{2}} x^{a-1}}{x-1}\, dx = \int_0^1 \frac{x^{a-1} - x^{-a}}{1-x}\, dx,$$

for $0 > a > 1$.

As the verification of (β) is involved in some needless complications, we shall be here contented with the discussion about the incorrectness of the formula (α) only.

We have

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \log \left(\frac{1+x}{1-x}\right) + \text{const.},$$

whence taken between the limits 0 and ∞,

$$\int_0^\infty \frac{dx}{1-x^2} = \frac{1}{2} \left\{ \log \left(\frac{1+x}{1-x}\right)_\infty - \log \left(\frac{1+x}{1-x}\right)_0 \right\}$$
$$= \frac{1}{2} \{\log(\infty) - \log(-\infty)\},$$

an expression which is altogether without meaning, — mathematics has nothing to do with such a meaningless formula.

Williamson has fallen into his mistake, because he paid little attention to the singularity in the problem, in spite of all that is very easy to see.

T. HAYASHI, ON THE NUMBER OF PRIME NUMBERS THAT ARE COMPRISED BETWEEN TWO GIVEN INTEGERS.[1])

We shall here give two formulae that represent the number of prime numbers that are comprised between two given whole numbers.

§ 1. The first of the formulae we are going to establish is a direct sequence of Laurent's theorem[2]), that finds its enunciation in these words:

1) Journal of Phys. Sch., Vol. 9, 1900.

2) H. Laurent, Sur les nombres premiers. Nouvelles Annales de Mathematiques. Tome XVIII. 1899.

If we designate

$$\begin{array}{l}(1-x)\,(1-x^2)\,(1-x^3)\dots(1-x^{n-1})\\(1-x^2)\,(1-x^4)\,(1-x^6)\dots(1-x^{2n-2})\\(1-x^3)\,(1-x^6)\,(1-x^8)\dots(1-x^{3n-3})\\\dots\dots\dots\dots\dots\dots\\(1-x^{n-1})\,(1-x^{2n-2})\,(1-x^{3n-3})\dots(1-x^{(n-1)^2})\end{array}$$

with $F_n(x)$, *then*

(1) $$F_n(\alpha)=0 \text{ or } F_n(\alpha)=n^{n-1},$$

where α *denotes an imaginary root of* $x^n-1=0$;

(2) $$F_n(x)=Q(x)\cdot\frac{x^n-1}{x-1}, \text{ or } = Q_1(x)\cdot\frac{x^n-1}{x-1}+n^{n-1},$$

where Q *and* Q_1 *are integral rational functions*;

(3) *the residuum of* $\frac{F_n(x)}{x^n-1}$ (*save for* $x=1$) $=0$, *or* $=-n^{n-2}$;

according as n *is not, or is, a prime number.*

This simple theorem, with that of Wilson, serves to distinguish between prime and non-prime numbers.

If $Rf(x)$ denote the residuum of $f(x)$, we have from Laurent's theorem

$$-R\frac{F_n(x)}{(x^n-1)\,n^{n-2}}=0, \text{ or } =1,$$

according as n is a non-prime or a prime number.

It therefore follows that the number of prime numbers comprised between two whole numbers s and t will be given by

$$-R\sum_{n=s}^{n=t}\frac{F_n(x)}{(x^n-1)\,n^{n-2}} \text{ or } -\frac{1}{2\pi}\sum_{n=s}^{n=t}\int_0^{2\pi}\frac{F_n\left(\gamma e^{\theta\sqrt{-1}}\right)\cdot\gamma e^{\theta\sqrt{-1}}\,d\theta}{n^{n-2}\left(\gamma^n e^{n\theta\sqrt{-1}}-1\right)}.$$

The sum of powers of these prime numbers raised to the same order is

$$-R\sum_{n=s}^{n=t}\frac{F_n(x)}{(x^n-1)\,n^{n-i-2}},$$

where i denotes the order of powers raised.

The sum of the reciprocals of these powers is

$$-R\sum_{n=s}^{n=t}\frac{F_n(x)}{(x^n-1)\,n^{n+i-2}}.$$

§ 2. Results similar to the above may be deduced by means of Wilson's theorem. Although Laurent has touched the theorem we have to give, yet our result is quite different from his.

Theorem. The roots of the equation $x^n - 1 = 0$ *being denoted by* $\alpha^i \{i = 0, 1, 2, 3, \ldots, (n-1)\}$, *we have*

$$\sum_{i=0}^{i=n-1} \alpha^{i\{(n-1)!+1\}} = n, \quad \text{or} \quad = 0,$$

according as n *is or is not a prime number.*

For, when n is a prime number, by Wilson's theorem we have

$$(n-1)! + 1 \equiv 0 \pmod{n}.$$

Hence for all values of i

$$\alpha^{i\{(n-1)!+1\}} = 1.$$

When n is a prime number, therefore,

$$\sum_{i=0}^{i=n-1} \alpha^{i\{(n-1)!+1\}} = n.$$

When n is not a prime number, we have also by Wilson's theorem

$$(n-1)! + 1 \equiv a \pmod{n},$$

where a is not only a positive integer, not equal to zero, but a and n are prime to each other; for had a and n a common factor, which that is not equal to unity, this common factor would be a factor of $(n-1)!$, and hence a factor of unity. Hence we have

$$i\{(n-1)! + 1\} \equiv ia \pmod{n};$$

so that

$$\sum_{i=0}^{i=n-1} \alpha^{i\{(n-1)!+1\}} = \sum_{i=0}^{i=n-1} \alpha^{ia}.$$

But the least positive residuum with respect to n of the series $0, a, 2a, \ldots, (n-1)a$, when a and n are mutually prime, is the same as that of the series

$$0, 1, 2, \ldots, (n-1).$$

Consequently

$$\begin{aligned}\sum_{i=0}^{i=n-1} \alpha^{i\{(n-1)!+1\}} &= \sum_{i=0}^{i=n-1} \alpha^{i} \\ &= 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} \\ &= 0,\end{aligned}$$

as we had to prove.

We next proceed from this theorem as follows:

Here we have

$$\sum_{i=0}^{i=n-1}(\alpha^i)^{(n-1)!+1} = nR\frac{x^{(n-1)!}}{x^{n-1}}$$

$$= \frac{n}{2\pi}\int_0^{2\pi}\frac{r^{(n-1)!+1}\,e^{\{(n-1)!+1\}\theta\sqrt{-1}}}{r^n e^{n\theta\sqrt{-1}}-1}\,d\theta,$$

$$(r>1).$$

Hence

$$R\frac{x^{(n-1)!}}{x^{n-1}} = 1 \quad \text{or} \quad 0,$$

according as n is or is not a prime number.

Therefore *the number of prime numbers that are comprised between s and t is*

$$R\sum_{n=s}^{n=t}\frac{x^{(n-1)!}}{x^n-1} \quad \textit{or} \quad \frac{1}{2\pi}\sum_{n=s}^{n=t}\int_0^{2\pi}\frac{r^{(n-1)!+1}\,e^{\{(n-1)!+1\}\theta\sqrt{-1}}}{r^n e^{n\theta\sqrt{-1}}-1}\,d\theta.$$

We also obtain a theorem that concerns to a definite integral. This theorem finds its expression in the formula

$$\int_0^{2\pi}\frac{r^{(n-1)!+1}\left[\cos\{(n-1)!+1\}\theta+\sqrt{-1}\sin\{(n-1)!+1\}\theta\right]d\theta}{r^n\left[\cos n\theta+\sqrt{-1}\sin n\theta\right]-1}$$

$= 2\pi$ (when n is a prime number),

or

$= 0$ (when n is not a prime number).

Here r is supposed to be >1.

T. HAYASHI, NEW THEOREMS ON THE INTEGRAL AND THE SPHERICAL FUNCTION.[1]

In the integral given at the end of the preceding paper, by transforming the denominator and separating the real and imaginary parts, we get

$$(1)\qquad \int_0^{2\pi} r^{m+n}\frac{\cos(m-n)\theta - r^m\cos m\theta}{1-2r^n\cos n\theta + r^{2n}}\,d\theta = 2\pi \quad \text{or} \quad 0,$$

as n is a prime number or not, and

1) The Journal of Phys. Sch. in Tokyo, Vol. 9, pp. 217—220, June, 1900.

$$(2)\qquad \int_0^{2\pi} \frac{r^{m+n}\sin(m-n)\theta + r^m \sin m\theta}{1-2r^n\cos n\theta + r^{2n}}\,d\theta = 0,$$

where $m = (n-1)! + 1$, $r > 1$, in both formulae.

In this place we shall prove these formulae in a way different from what has been said in the former paper.

§ 1. Changing in (1) r into $\frac{1}{r}$ the integral transforms into

$$\int_0^{2\pi} \frac{\{\cos(m-n)\theta - r^n\cos m\theta\}r^{n-m}}{1-2r^n\cos n\theta + r^{2n}}\,d\theta \qquad (r<1).$$

But, as is well-known, for $\alpha < 1$, we have

$$\frac{1-\alpha^2}{1-2\alpha\cos\gamma+\alpha^2} = 1 + 2\alpha\cos\gamma + 2\alpha^2\cos 2\gamma + \ldots$$
$$= 1 + 2\sum_{p=1}^{\infty}\alpha^p\cos p\gamma,$$

so that, r being < 1, we get

$$\frac{1}{1-2r^n\cos n\theta + r^{2n}} = \frac{1}{1-r^{2n}}\left\{1 + 2\sum_{p=1}^{\infty} r^{pn}\cos pn\theta\right\}.$$

Hence (1) becomes

$$\frac{r^{n-m}}{1-r^{2n}}\int_0^{2\pi}\left\{\cos(m-n)\theta - r^n\cos m\theta\right\}\left\{1+2\sum_{p=1}^{\infty} r^{pn}\cos pn\theta\right\}d\theta.$$

We have evidently

$$\int_0^{2\pi}\cos s\theta\,d\theta = 0 \qquad (s \neq 0).$$

We therefore distinguish the cases $n = 2$ and $n \neq 2$.

If we make $n = 2$, then

$$m = (n-1)! + 1 = 2,$$

so that the integral assumes the form

$$\frac{1}{1-r^4}\left[2\pi - r^2\sum_{p=1}^{\infty} r^{2p}\int_0^{2\pi} 2\cos 2\theta\cos 2p\theta\,d\theta\right].$$

But in general

$$\int_0^{2\pi} 2\cos s\theta\cos t\theta\,d\theta = 2\pi \qquad (s=t),$$

or

$$= 0 \qquad (s \neq t).$$

Hence

$$\sum_{p=1}^{\infty} r^{2p} \int_0^{2\pi} 2 \cos 2\theta \cos 2p\theta \, d\theta = 2\pi r^2.$$

The value of (1) is therefore equal to 2π for $n = 2$.

For $n \neq 2$, we have

$$\frac{r^{n-m}}{1-r^{2n}} \sum_{p=1}^{\infty} r^{pn} \int_0^{2\pi} 2 \{\cos (m-n)\theta - r^n \cos m\theta\} \cos pn\theta \, d\theta.$$

But in general

$$\begin{aligned} \int_0^{2\pi} 2 \cos s\theta \cos t\theta \, d\theta &= 2\pi \qquad (s = t), \\ &= 0 \qquad (s \neq t). \end{aligned}$$

If n is a prime number we can put

$$m - n = (n-1)! + 1 - n = p_1 n,$$

$$m = (n-1)! + 1 = p_2 n,$$

so that, writing $(n-1)! + 1 = An$, we have

$$p_1 = A - 1, \quad p_2 = A,$$

and the integral becomes

$$\begin{aligned} \frac{r^{n-m}}{1-r^{2n}} \left[r^{(A-1)n} 2\pi - r^{An} r^n 2\pi\right] &= 2\pi \frac{r^{n-m}}{1-r^{2n}} \left[r^{(A-1)n} - r^{(A+1)n}\right] \\ &= 2\pi \frac{r^{n-m}}{1-r^{2n}} \left[r^{m-n} - r^{m+n}\right] \\ &= 2\pi. \end{aligned}$$

When n is a non-prime number, we cannot find the values of p_1 and p_2 such that $m - n = p_1 n$, $m = p_2 n$. Hence the integral is equal to zero in this case.

In the same way we see that the integral (2) is $= 0$.

§ 2. From the above formulae, theorems on the spherical function may be derived.

Writing

$$\frac{1}{1 - 2\alpha \cos \gamma + \alpha^2} = \sum_{q=0}^{\infty} Q_q(\gamma) \alpha^q,$$

(1) becomes

$$r^{n-m} \sum_{q=0}^{\infty} r^{qn} \int_0^{2\pi} \{\cos (m-n)\theta - r^n \cos m\theta\} Q_q(n\theta) \, d\theta.$$

For $n=2$,

$$\sum_{q=0}^{\infty} r^{2q} \int_0^{2\pi} \{1-r^2\cos 2\theta\} Q_q(2\theta)\, d\theta = 2\pi,$$

or

$$\sum_{q=0}^{\infty} r^{2q} \int_0^{2\pi} Q_q(2\theta)\, d\theta - \sum_{q=1}^{\infty} r^{2q} \int_0^{2\pi} \cos 2\theta\, Q_{q-1}(2\theta)\, d\theta = 2\pi.$$

But evidently

$$\int_0^{2\pi} Q_0(2\theta)\, d\theta = 2\pi,$$

so that

$$\int_0^{2\pi} \{Q_q(2\theta) - \cos 2\theta\, Q_{q-1}(2\theta)\}\, d\theta = 0,$$

that is,

$$(3) \qquad \int_0^{2\pi} Q_q(2\theta)\, d\theta = \int_0^{2\pi} \cos 2\theta\, Q_{q-1}(2\theta)\, d\theta \qquad (q \neq 0).$$

Next we take n for a prime number other than 2; then

$$r^{n-m}\Big[\int_0^{2\pi} \cos(m-n)\theta \cdot Q_0(n\theta)\, d\theta$$
$$+\sum_{q=1}^{\infty} r^{qn} \int_0^{2\pi} \{\cos(m-n)\theta \cdot Q_q(n\theta) - \cos m\theta \cdot Q_{q-1}(n\theta)\}\, d\theta\Big] = 2\pi.$$

Hence if we take the value of q that satisfies

$$m-n=(n-1)!+1-n=qn \qquad (q \neq 0),$$

we shall have

$$(4) \quad \int_0^{2\pi} \cos(m-n)\theta \cdot Q_q(n\theta)\, d\theta = 2\pi + \int_0^{2\pi} \cos m\theta \cdot Q_{q-1}(n\theta)\, d\theta.$$

For other values of q we have

$$(5) \qquad \int_0^{2\pi} \cos(m-n)\theta \cdot Q_q(n\theta)\, d\theta = \int_0^{2\pi} \cos m\theta \cdot Q_{q-2}(n\theta)\, d\theta.$$

Since it is evidently

$$\int_0^{2\pi} \cos(m-n)\theta \cdot Q_0(n\theta)\, d\theta = 0,$$

we shall have for any value of q, when n is a non-prime number,

$$(6) \qquad \int_0^{2\pi} \cos(m-n)\theta \cdot Q_q(n\theta)\, d\theta = \int_0^{2\pi} \cos m\theta \cdot Q_{q-1}(n\theta)\, d\theta.$$

From (2) by a similar reasoning we device, for any value of n and q, the formula

$$(7) \quad \int_0^{2\pi} \sin(m-n)\theta \cdot Q_q(n\theta)\, d\theta + \int_0^{2\pi} \sin m\theta \cdot Q_{q-1}(n\theta)\, d\theta = 0.$$

T. HAYASHI, ON A THEOREM OF ABEL THAT CONCERNS TO THE EXPANSION OF FUNCTIONS.[1])

Here we propose a study on the following series

$$(1)\qquad \Phi(x+a)=\Phi(x)+a\Phi'(x+b)+\frac{a(a-2b)}{2!}\Phi''(x+2b)+\ldots$$
$$+\frac{a(a-rb)^{r-1}}{r!}\Phi^{(r)}(x+rb)+\ldots$$

This series in its general form is said not to hold in many cases. Bertrand however succeeded by an application of Biermann's formula in establishing the series for the case when the function $\Phi(x)$ is capable of being expanded in an ascending power series of e^x. Besides this case with such a condition, we seldom meet with any application of the series. We have therefore to investigate the question, under what conditions the series will hold, and in case such a condition is required to exist, to see from what circumstance that condition comes to enter into consideration.

In order that the series (1) should hold, the function $\Phi(x)$ must be capable of an expansion by Taylor's theorem, for the series (1) becomes identical with Taylor's expansion for $b=0$. Thus we have

$$(2)\qquad \Phi(x+a)=\Phi(x)+a\Phi'(x)+\frac{a^2}{2!}\Phi''(x)+\ldots$$

By rearranging the terms in (1) according to the ascending powers of a, we get

$$(3)\quad \Phi(x+a)=\Phi(x)+\sum_{m=1}^{\infty}\frac{a^m}{m!}\sum_{r-1}^{\infty}\left\{\frac{(-1)^{r-1}m(r+m-1)^{r-2}b^{r-1}}{(r-1)!}\right.$$
$$\left.\times\,\Phi^{(r+m-1)}(x+\overline{r+m-1}\,b)\right\}.$$

The three series (1), (2) and (3) must be all convergent and must have the equal sum. Hence by comparison of coefficients we get

$$(4)\qquad \Phi^{(m)}(x)=\sum_{r=1}^{\infty}(-1)^{r-1}\frac{m(m+r-1)^{r-2}b^{r-1}}{(r-1)!}$$
$$\times\,\Phi^{(m+r-1)}(x+\overline{m+r-1}\,b).$$

If in this expression we expand $\Phi^{(m+r-1)}(x+\overline{m+r-1}\,b)$ and arrange the result in order of the powers of b, the coefficients of these powers of b must all identically vanish for all values of m. Thus

1) Journal of Tokyo Phys. Sch., Vol. 9, pp. 353—355 and 387—390, Oktober and November, 1900.

$$\Phi^{(m)}(x) = m\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(-1)^{r-1}\cdot\frac{(m+r-1)^{r+s-3}}{(r-1)!\,(s-1)!}\cdot\Phi^{(m+r+s-2)}(x)\,b^{r+s-2}$$

$$= m\sum_{r=1}^{\infty}\sum_{\varrho=r-1}^{\infty}(-1)^{r-1}\cdot\frac{(m+r-1)^{\varrho-1}}{(r-1)!\,(\varrho-r+1)!}\cdot\Phi^{(m+\varrho)}(x)\,b^{\varrho}$$

$$= \sum_{t=0}^{\infty}\left[\Phi^{(m+t)}(x)\,b^{t}\sum_{\tau=0}^{t}(-1)^{\tau}\cdot\frac{(m+\tau)^{t-1}}{\tau!\,(t-\tau)!}\right],$$

so that for all values of m and t we must have

$$(5)\qquad \sum_{\tau=0}^{t}(-1)^{\tau}\frac{(m+\tau)^{t-1}}{\tau!\,(t-\tau)!} = 0.$$

But we have

$$\sum_{\tau=0}^{n}\frac{n!}{\tau!\,(n-\tau)!}(a-\tau b)^{p} = 0 \qquad (p<n),$$

as S. Nakagawa gives in his paper published in the *Journal of the Tokyo Physics School.*

In this equality put $b=-1$, $p=n-1$, $n=t$, and then divide by $t!$; and thus we see the formula (5) established.

In this way the series of Abel's that stands before us has been rendered applicable in far wider a scope than settled by Bertrand. In the above reasoning we have sometimes changed a singly infinite series into an infinite double series and vice versa. The condition for applicability would have probably arisen from such a circumstance.

Whatever be the restriction on such a scope, the series (1) does not lack of a sound establishment.

By adding

$$-b\sum_{r=1}^{\infty}\frac{(a-rb)^{r-1}}{(r-1)!}\,\Phi^{(r)}(x+rb)$$

to both sides of (1), we have

$$\Phi(x+a) - b\sum_{r=1}^{\infty}\frac{(a-rb)^{r-1}}{(r-1)!}\,\Phi^{(r)}(x+rb) = \sum_{r=0}^{\infty}\frac{(a-rb)^{r}}{r!}\,\Phi^{(r)}(x+rb),$$

or, as we have to put,

$$(6)\qquad \Psi(x,a) = \sum_{r=0}^{\infty}\frac{(a-rb)^{r}}{r!}\,\Phi^{(r)}(x+rb).$$

From this follows

$$(7) \qquad \frac{d}{dx} \Psi(x+b),\, a-b) = \sum_{r=1}^{\infty} \frac{(a-rb)^{r-1}}{(r-1)!} \Phi^{(r)}(x+rb).$$

And hence we obtain the relation

$$(8) \qquad \Phi(x+a) - b\frac{d}{dx}\Psi(x+a,\, a-b) = \Psi(x,\, a).$$

The series in (6) and (7) are both symmetrical with respect to x and a. This fact will be proved by rearranging these series in the double series arranged in order of powers of a and b and changing the coefficients of b according to the formula

$$\sum_{r=1}^{\infty} (-1)^r \frac{(a-rb)^n}{r!(n-r)!} = b^n,$$

and then paying attention to those of a.

The series in (6) and (7) are not only symmetrical with respect to x and a, but they are both functions of $(x+a)$, namely, the former is equal to

$$\Phi(x+a) - b\Phi'(x+a) + b^2\Phi''(x+a) - b^3\Phi'''(x+a) + \ldots,$$

while the latter is equal to

$$\Phi'(x+a) - b\Phi''(x+a) + b^2\Phi'''(x+a) - \ldots$$

We here apply the method of the *Calcul de Généralisation* given by Oltramaré of Geneva to solve the differential equation (8) and to determine the form of the function $\Psi(x,\, a)$. This method might appear too artificial and too symbolical, but it is at the same time very convenient for solving an equation of the kind described.

From (8) we have

$$\Psi(x,\, a) + b\frac{d}{dx}\Psi(x+a, a-b) = \Phi(x+b).$$

Now writing

$$Ge^{xu+av} = \Psi(x,a), \quad Ge^{xu'+av'} = \Phi(x+a),$$

where G stands for the symbol of generalisation, the last written equation will be rewritten in the form

$$Ge^{xu+av}\{1 + bue^{b(u-v)}\} = Ge^{xu'+av'}.$$

$$\therefore\ Ge^{xu+av} = G\frac{e^{xu'+av'}}{1+bu'e^{b(u'-v')}}$$
$$= Ge^{xu'+av'}\{1 - bu'e^{b(u'-v')} + b^2u'^2e^{2b(u'-v')^2} - + \dots\}$$
$$= Ge^{xu'+av'} - bGu'e^{(x+b)u'+(a-b)v'}$$
$$+ b^2Gu'^2e^{(x+2b)u'+(a-2b)v'} - + \dots$$
$$\therefore\quad \Psi(x, a) = \Phi(x+a) - b\Phi'(x+a) + b^2\Phi''(x+a) - + \dots$$

See how simple this process is!

Abel's series affords in its general form a rich source for identities; for a and b are arbitrary and x may have any value too, that lies within the domain for convergency.

It is noteworthy that the expressions we have employed in the above reasoning, and also those that will be obtained by transforming the series before us will also constitute a large stock for identities.

Thus in (4) replacing $\Phi(x)$ in place of $\Phi^{(m)}(x)$ and $r+1$ for r, we get

$$(9)\qquad \Phi(x) = \sum_{r=0}^{\infty}(-1)^r\frac{m(m+r)^{r-1}b^r}{r!}\Phi^{(r)}(x+\overline{m+r}\,b),$$

where m is to be any positive integer, but may finally assume any value whatever as will be seen from what we have reasoned about.

Put $x=0$ in (9) and replace b by x, then we get

$$\Phi(0) = \sum_{r=0}^{\infty}(-1)^r\frac{m(m+r)^{r-1}x^r}{r!}\Phi^{(r)}(\overline{m+r}\,b).$$

Put $m=1$ in the same formula, when we get

$$\Phi(x) = \sum_{r=0}^{\infty}(-1)^r\frac{(r+1)^{r-1}b^r}{r!}\Phi^{(r)}\{x+(1+r)b\};$$

and the sign of b being altered,

$$\Phi(x) = \sum_{r=0}^{\infty}\frac{(r+1)^{r-1}b^r}{r!}\Phi^{(r)}\{x-(1+r)b\};$$

and then write $x+b$ in place of x and we shall have

$$\Phi(x+b) = \sum_{r=0}^{\infty}(r+1)^{r-1}\frac{b^r}{r!}\Phi^{(r)}(x-rb).$$

Again in (9) we replace mb by a and get

$$\Phi(x) = \sum_{r=0}^{\infty} (-1)^r \frac{a(a+rb)^{r-1}}{r!} \Phi^{(r)}(x+a+rb);$$

$(x-a)$ being put for x,

$$\Phi(x-a) = \sum_{r=0}^{\infty} (-1)^r \frac{a(a+rb)^{r-1}}{r!} \Phi^{(r)}(x+rb);$$

the sign of a being changed

$$\Phi(x+a) = \sum_{r=0}^{\infty} \frac{a(a-rb)^{r-1}}{r!} \Phi^{(r)}(x+rb),$$

which is the very series of Abel's from which we have started. Thus it will be seen that Abel's series itself is contained in the series (4).

T. HAYASHI, AN EXAMPLE OF FINDING A PARTICULAR SOLUTION OF AN EQUATION THAT WILL BE SATISFIED BY ABEL'S SYMMETRICAL FUNCTION.[1]

What we call in this place by the name of Abel's symmetrical function is one that was considered by Abel in the first volume of Crelle's Journal, where we see the proposition proved:

If a function of two variables, $\Psi(x, y)$, be such that $\Psi\big(\Psi(x, y), z\big)$ is a symmetrical function of x, y, z, so there will always exist a functional form $\Phi(u)$ that satisfies

$$\Phi\Psi(x, y) = \Phi(x) + \Phi(y).$$

About this subject the author has considered in the *Journal of the Mathematico-Physical Society in Tokyo*, Vol. 8, Part 4. But the theorem may be evidently extended and we get the following proposition:

When

$$\Psi_n(a_1, a_2, \ldots, a_n) = \Psi\big(\Psi_{n-1}(a_1, a_2, \ldots, a_{n-1}), a_n\big)$$

is a symmetrical function of $a_1, a_2, \ldots, a_n$, then there will always exist a functional form $\Phi(u)$ that satisfies

$$\Phi\Psi_n(a_1, a_2, \ldots, a_n) = \Phi(a_1) + \Phi(a_2) + \ldots + \Phi(a_n).$$

1) Journal of Tokyo Physics School, Vol. 11, pp. 80—84, February, 1902. For the conceptions in the present paper the author acknowledges his indebtedness to Lémeray (Nouvelles Annales de Mathematiques, N. S., Tome 1).

If $\Psi_n(a_1, a_2, \ldots, a_n)$ *have a definite meaning when* n *assumes an indefinitely great value, and if*

$$\lim_{n=\infty} \Psi_n(a_1, a_2, \ldots, a_n)$$

be denoted by $\Psi_\infty(a_1, a_2, \ldots, a_n)$, *then since*

$$\lim_{n=\infty} \Phi\Psi_n(a_1, a_2, \ldots, a_n) = \lim_{n=\infty}\{\Phi(a_1) + \Phi(a_2) + \ldots + \Phi(a_n)\},$$

there may happen such a functional form $\Phi(u)$ *to exist that satisfies*

$$\Phi\Psi_\infty(a_1, a_2, a_3, \ldots) = \Phi(a_1) + \Phi(a_2) + \Phi(a_3) + \ldots$$

For instance, if we take

$$a_n = \frac{1}{x+n},$$

$$\Psi_n(a_1, a_2, \ldots, a_n) = \sqrt[s]{\frac{1}{(x+1)^s} + \frac{1}{(x+2)^s} + \ldots + \frac{1}{(x+n)^s}},$$

then Ψ_∞ will have a finite value for $x > 0$ and $s > 1$. And if we make $\Phi(u) = u^s$, we shall have

$$\Phi\Psi_\infty = \frac{1}{(x+1)^s} + \frac{1}{(x+2)^s} + \frac{1}{(x+3)^s} + \ldots,$$

and

$$\Phi(a_1) + \Phi(a_2) + \ldots = \frac{1}{(x+1)^s} + \frac{1}{(x+2)^s} + \ldots$$

Let $F(x)$ be a given function and let a be a positive integer. Then the function $f(x)$ that satisfies the functional equation

$$f(x)\, f(x+a)\, f(x+2a) \ldots f(x+\overline{m-1}\,a) = F(x)$$

will be seen easily to have the form

$$f(x) = \prod_{n=0}^{\infty} \frac{F(x+mna)}{F\{x+(mn+1)a\}} \cdot \prod_{k=1}^{am-1} C_k^{\frac{2k\pi}{am}\sqrt{-1}\cdot x},$$

where the C's are arbitrary constants. (See the *Calcul de Généralisation.*) Hence

$$f(x) = \prod_{n=0}^{\infty} \frac{F(x+mna)}{F\{x+(mn+1)a\}}$$

is a particular solution.

Hence a function $f(x)$, that satisfies

$$\log f(x) + \log f(x+a) + \log f(x+2a) + \ldots + \log f(x+\overline{m-1}\,a) = \log F(x),$$

is of the form

$$\log f(x) = \sum_{n=0}^{\infty} \log F(x+mna) - \sum_{n=0}^{\infty} \log\{x+(mn+1)a\}\cdot$$

By replacing $\log f(x)$ by $f(x)$, and $\log F(x)$ by $F(x)$, we arrive at the result:

A particular solution of the functional equation

$$f(x)+f(x+a)+f(x+2a)+\ldots+f\{x+(m-1)a\}=F(x),$$

where $F(x)$ is a given function, is of the form

$$f(x)=\sum_{n=0}^{\infty}F(x+amn)-\sum_{n=0}^{\infty}F\{x+a(mn+1)\}.$$

The sum of the terms in the left-hand member of this functional equation seems in some cases to be replaceable by Abel's symmetrical function of these terms. In other words,

A functional form $f(x)$ that satisfies the equation

$$\Psi_m\{f(x),\ f(x+a),\ f(x+2a),\ldots,f(x+\overline{m-1}\,a)\}=F(x),$$

where Ψ_m is Abel's symmetrical function, may be found.

As $\Psi_m(a_1, a_2, \ldots, a_m)$ is Abel's symmetrical function, there will exist a functional form $\Phi(u)$ that satisfies

$$\Phi\Psi_m(a_1, a_2, \ldots, a_m)=\Phi(a_1)+\Phi(a_2)+\ldots+\Phi(a_m).$$

Let this $\Phi(u)$ be such that satisfies

$$\Phi\Psi_\infty(a_1, a_2, \ldots)=\Phi(a_1)+\Phi(a_2)+\ldots;$$

then by putting $u=\Psi_2(v,w)$, we have $\Phi(u)=\Phi(v)+\Phi(w)$, and hence $\Phi(w)=\Phi(u)-\Phi(v)$. Hence if we solve $u=\Psi_2(v,w)$ for w and put $w=\Omega(u,v)$, we have

$$\Phi\{\Omega(u,v)\}=\Phi(u)-\Phi(v).$$

Writing $f(x), f(x+a), \ldots, f(x+\overline{m-1}a)$ in place of $a_1, a_2, \ldots, a_m$, we have

$$\Phi F(x)=\Phi f(x)+\Phi f(x+a)+\Phi f(x+2a)+\ldots+\Phi f(x+\overline{m-1}\,a).$$

Hence from the particular solution above obtained we get

$$\begin{aligned}\Phi f(x)&=\Phi\Psi_\infty\{F(x),\ F(x+ma),\ F(x+2ma),\ldots\}\\&\quad-\Phi\Psi_\infty\{F(x+a),\ F(x+\overline{m+1}\,a),\ F(x+\overline{2m+1}\,a),\ldots\}.\end{aligned}$$

Hence putting

$$\Psi_\infty'(x)=\Psi_\infty\{F(x),\ F(x+ma),\ F(x+2ma),\ldots\},$$

we obtain

$$\Phi f(x)=\Phi\Psi_\infty'(x)-\Phi\Psi_\infty'(x+a)=\Phi\Omega\{\Psi_\infty'(x),\ \Psi_\infty'(x+a)\},$$

whence again follows

$$f(x)=\Omega\{\Psi_\infty'(x),\ \Psi_\infty'(x+a)\}.$$

Therefore $F(x)$ *being a given function and* Ψ_m *Abel's symmetrical function, a particular solution of the equation*

$$\Psi_m\{f(x),\ f(x+a),\ f(x+2a),\ \ldots,\ f(x+\overline{m-1}\,a)\} = F(x)$$

is of the form

$$f(x) = \Omega\{\Psi'_\infty(x),\ \Psi'_\infty(x+a)\},$$

where $w = \Omega(u, v)$ *is got by solving* $u = \Psi_2(v, w)$, *and where*

$$\Psi'_\infty(x) = \Psi_\infty\{F(x),\ F(x+ma),\ F(x+2ma), \ldots\}.$$

Example 1. To solve the functional equation

$$\frac{f(x)+f(x+a)}{1+f(x)\,f(x+a)} = F(x).$$

Here we have, by writing

$$\{1 \pm F(x)\}\{1 \pm F(x+2a)\}\{1 \pm F(x+4a)\} \ldots = A \text{ and } B,$$

where the upper and lower signs are assigned to A and B respectively,

$$\Psi'_\infty(x) = \frac{A-B}{A+B},$$

and

$$u = \frac{v+w}{1+vw}, \quad \therefore\ w = \frac{u-v}{1-uv} = \Omega(u, v),$$

$$\therefore\ f(x) = \frac{\Psi'_\infty(x) - \Psi'_\infty(x+a)}{1-\Psi'_\infty(x)\,\Psi'_\infty(x+a)}.$$

Example 2. To find the function that satisfies

$$f(x)^m + f(x+a)^m = \frac{1}{x^{2m}},$$

where m *is a positive integer.*

Here $\sqrt[m]{f(x)^m + f(x+a)^m}$ is Abel's symmetrical function, and we have

$$\Psi'_\infty(x) = \sqrt[m]{\left\{\frac{1}{x^{2m}} + \frac{1}{(x+2a)^{2m}} + \frac{1}{(x+4a)^{2m}} + \ldots\right\}},$$

$$u = \sqrt[m]{v^m + w^m}, \quad \therefore\ w = \sqrt[m]{u^m - v^m} = \Omega(u, v),$$

$$\therefore\ f(x) = \sqrt[m]{\left\{\frac{1}{x^{2m}} - \frac{1}{(x+a)^{2m}} + \frac{1}{(x+2a)^{2m}} - \ldots\right\}}.$$

M. KABA, ON THE MULTIPLICATION OF THE ELLIPTICAL FUNCTION.[1]

The formulae for $\operatorname{cn} mu$ and $\operatorname{dn} mu$ resemble to the expression for $\operatorname{sn} mu$, so that we shall speak only about the last named function, and indeed only for the case where m is an odd whole number, for the case of m even could be derived without encountering any difficulty from the other case.

The present study will be carried in the order as shown below:

I. *To express the numerator and denominator in the expansion of* $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ *in the form of products of binomial factors upon the consideration of its poles and zero-points.*

II. *The expansion of the denominator will be made out as soon as that for the numerator is known.*

III. *To express the numerator in a power series of* $\operatorname{sn} u$.

IV. *To express the denominator in a power series of* $\operatorname{sn} u$.

V. *To express the numerator by a determinant.*

VI. *To express the denominator by a determinant.*

§ 1. As m is an odd number, $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ is an even function of u. Hence expressed in terms of $\operatorname{sn} u$ which is an odd function, it must be a function of $\operatorname{sn}^2 u$. If then 4ω and $2\omega'$ denote the periods of $\operatorname{sn} u$, the function $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ will have

$$\text{the periods } 4\omega \text{ and } 2\omega';$$

and

$$\text{the zero points } \frac{2n\omega + 2n'\omega'}{m},$$

where

$$n, n' = 0, 1, 2, \ldots, (m-1),$$

the simultaneous values $n = 0$, $n' = 0$ being excluded; and

$$\text{the poles } \frac{2n\omega + 2n'\omega' + \omega'}{m},$$

where n and n' have the same significations as before (the same will be meant in the following), the simultaneous values $n = 0$, $n' = \frac{m-1}{2}$ being excluded.

1) The Journal of the Physics School in Tokyo, Vol. 10, 1901.

Thus the function $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ is expressible in terms of $\operatorname{sn} u$ that possesses the periods 4ω and $2\omega'$, namely,

$$\frac{\operatorname{sn} mu}{\operatorname{sn} u} = C \cdot \frac{\prod\left(\operatorname{sn} u - \operatorname{sn} \frac{2n\omega + 2n'\omega'}{m}\right)}{\prod\left(\operatorname{sn} u - \operatorname{sn} \frac{2n\omega + 2n'\omega' + \omega'}{m}\right)}.$$

The numerator and denominator in this expression are of degrees $m^2 - 1$. The constant C may be expressed by putting $u = \omega'$; that is

$$\left(\frac{\operatorname{sn} mu}{\operatorname{sn} u}\right)_{u=\omega'} = \left(\frac{\operatorname{sn} m(u+\omega')}{\operatorname{sn}(u+\omega')}\right)_{u=0} = \left(\frac{\operatorname{sn}(mu + m\omega')}{\operatorname{sn}(u+\omega')}\right)_{u=0} = \left(\frac{\operatorname{sn} u}{\operatorname{sn} mu}\right)_{u=0} = \frac{1}{m}.$$

Therefore

$$\frac{\operatorname{sn} mu}{\operatorname{sn} u} = \frac{\prod\left(\operatorname{sn} u - \operatorname{sn} \frac{2n\omega + 2n'\omega'}{m}\right)}{m\prod\left(\operatorname{sn} u - \operatorname{sn} \frac{2n\omega + 2n'\omega' + \omega'}{m}\right)}.$$

The numerator and denominator will be of the $(m^2-1)^{\text{th}}$ degree when expressed in the form of polynomials.

§ 2. Here we consider the function

$$\Phi_m = \frac{\sigma_3(mv)}{\sigma_3(v)^{m^2}},$$

where v corresponds to $\frac{u}{\sqrt{e_1 - e_3}}$ and e_1 and e_2 to $\wp(\omega_1)$ and $\wp(\omega_3)$ Then we have the correspondences $\omega_1 = \frac{\omega}{\sqrt{e_1 - e_3}}$, $\omega_3 = \frac{\omega'}{\sqrt{e_1 - e_3}}$.

The function Φ_m has

the periods $4\omega_1$ and $2\omega_3$;

the zero points $\frac{2n\omega_1 + 2n'\omega_3 + \omega_3}{m}$,

the pair of values $n = 0$, $n' = \frac{m-1}{2}$ being excluded; and

the pole of the $(m^2-1)^{\text{th}}$ order ω_3.

Hence the function may be expressed as a product of binomial factors containing $\operatorname{sn} u$; namely,

$$\frac{\sigma_3(mv)}{\sigma_3(v)^{m^2}} = C\prod\left(\operatorname{sn} u - \operatorname{sn} \frac{2n\omega + 2n'\omega' + \omega'}{m}\right).$$

The constant C will be determined in this manner. Divide both sides by $(\operatorname{sn} u)^{m^2-1}$ and put $v = \omega_3$, $u = \omega'$, and we have

$$C = \left(\frac{\sigma_3(mv)}{\sigma_3(v)\,(\operatorname{sn} u)^{m^2-1}\,\sigma_3(v)}\right)_{u=\omega',\ v=\omega_3}.$$

But

$$\operatorname{sn} u = \sqrt{e_1 - e_3} \cdot \frac{\sigma(v)}{\sigma_3(v)},$$

$$\therefore \ (\sigma_3(v) \operatorname{sn} u)_{u=\omega',\, v=\omega_3} = (\sigma(v) \sqrt{e_1 - e_3})_{v=\omega_3} = \sigma(\omega_3) \sqrt{e_1 - e_3};$$

and

$$\left(\frac{\sigma_3(mv)}{\sigma_3(v)}\right)_{v=\omega_3} = \left(\frac{\sigma_3\{m(v+\omega_3)\}}{\sigma_3(v+\omega_3)}\right)_{v=0} = (-1)^{\frac{m-1}{2}} \left(e^{\frac{\eta_3\omega_3}{2}}\right)^{m^2-1},$$

where $\zeta(\omega_1) = \eta_1$, $\zeta(\omega_2) = \eta_2$, $\zeta(\omega_3) = \eta_3$. Hence

$$C = (-1)^{\frac{m-1}{2}} m \left(\frac{e^{\frac{\eta_3\omega_3}{2}}}{\sigma(\omega_3)\sqrt{e_1 - e_3}}\right)^{m^2-1} = (-1)^{\frac{m-1}{2}} m \sqrt{k}^{\,m^2-1},$$

where $k^2 = \dfrac{e_2 - e_3}{e_1 - e_3}$.

We have therefore

$$\frac{\sigma_3(mv)}{\sigma_3(v)^{m^2}} = (-1)^{\frac{m-1}{2}} m \sqrt{k}^{\,m^2-1} \prod \left(\operatorname{sn} u - \operatorname{sn} \frac{2n\omega + 2n'\omega' + \omega'}{m}\right).$$

Now multiply $(-1)^{\frac{m-1}{2}} \sqrt{k}^{\,m^2-1}$ to the numerator and denumerator of the expression of $\dfrac{\operatorname{sn} mu}{\operatorname{sn} u}$ obtained in § 1, and we have

$$\text{(A)} \qquad \frac{\operatorname{sn} mu}{\operatorname{sn} u} = \frac{(-1)^{\frac{m-1}{2}} \sqrt{k}^{\,m^2-1} \prod \left(\operatorname{sn} u - \operatorname{sn} \dfrac{2n\omega + 2n'\omega'}{m}\right)}{(-1)^{\frac{m-1}{2}} m \sqrt{k}^{\,m^2-1} \prod \left(\operatorname{sn} u - \operatorname{sn} \dfrac{2n\omega + 2n'\omega' + \omega'}{m}\right)}.$$

The denominator is equal to $\dfrac{\sigma_3(mv)}{\sigma_3(v)^{m^2}}$, and the numerator to

$$\frac{\sigma_3(mv)}{\sigma_3(v)^{m^2}} \cdot \frac{\operatorname{sn} mu}{\operatorname{sn} u} = \frac{\sigma_3(mv)}{\sigma_3(v)^{m^2}} \cdot \frac{\sigma(mv)\sqrt{e_1 - e_3}}{\sigma_3(mv) \operatorname{sn} u} = \frac{\sigma(mv)}{\sigma_3(v)^{m^2}} \cdot \frac{\sqrt{e_1 - e_3}}{\operatorname{sn} u}.$$

But this last quantity is equal to the product of the denominator, where u will be changed into $u + \omega'$, that is, v changed into $v + \omega_3$, namely,

$$\frac{\sigma_3(m(v+\omega_3))}{\sigma_3(v+\omega_3)^{m^2}} = (-1)^{\frac{m-1}{2}} \frac{\sigma(mv)}{\sigma(v)^{m^2}} \left(\frac{\sqrt{k}}{\sqrt{e_1 - e_3}}\right)^{m^2-1},$$

multiplied with

$$M = (-1)^{\frac{m-1}{2}} \left(\frac{\sigma(v)}{\sigma_3(v)}\right)^{m^2} \left(\frac{\sqrt{e_1 - e_3}}{\sqrt{k}}\right)^{m^2-1} \frac{\sqrt{e_1 - e_3}}{\operatorname{sn} u} = (-1)^{\frac{m-1}{2}} (\sqrt{k} \operatorname{sn} u)^{m^2-1}.$$

Hence if we put $\operatorname{sn} u = x$ and represent the denominator and numerator in the expansion of $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ by $D(x^2)$ and $A(x^2)$ respectively, we have

$$\operatorname{sn}(u + \omega') = \frac{1}{k \operatorname{sn} u} = \frac{1}{kx},$$

$$A(x^2) = (-1)^{\frac{m-1}{2}} (\sqrt{k}\, x)^{m^2-1} D\left(\frac{1}{k^2 x^2}\right).$$

Again we put

$$\frac{\operatorname{sn} mu}{\operatorname{sn} u} = \frac{A_0 + A_2 x^2 + \ldots + A_{m^2-3} x^{m^2-3} + A_{m^2-1} x^{m^2-1}}{D_0 + D_2 x^2 + \ldots + D_{m^2-3} x^{m^2-3} + D_{m^2-1} x^{m^2-1}},$$

and compare with (A), when we obtain

$$D_{m^2-1} = (-1)^{\frac{m-1}{2}} m \sqrt{k}^{m^2-1},$$

$$A_{m^2-1} = (-1)^{\frac{m-1}{2}} m \sqrt{k}^{m^2-1}.$$

But

$$A_{m^2-1} x^{m^2-1} = (-1)^{\frac{m-1}{2}} m (\sqrt{k}\, x)^{m^2-1} D_0,$$

from what we have said above, so that D_0 is $= 1$.

If we make $u = 0$, then we get

$$\frac{A_0}{D_0} = m, \text{ or } A_0 = m D_0 = m.$$

§ 3. If we now denote by D the operation

$$D = 12 g_3 \frac{\partial}{\partial g_2} + \frac{2}{3} g_2 \frac{\partial}{\partial g_3},$$

and express $\wp(v)$ and $\zeta(v)$ by $\wp$ and ζ, we have

$$D\wp = 2\wp'\zeta + 4\wp^2 - \frac{2}{3} g_2,$$

$$De_\alpha = 4e_\alpha^2 - \frac{2}{3} g_2,$$

$$D\sigma_\alpha = \sigma_\alpha'' + \left(e_\alpha + \frac{1}{12} g_2 v^2\right) \sigma_3.$$

Since

$$\operatorname{sn} u = \sqrt{\frac{e_1 - e_3}{\wp - e_3}},$$

we have

$$D \operatorname{sn} u = D\sqrt{e_1 - e_3}\,\frac{1}{\sqrt{\wp - e_3}} + D\,\frac{1}{\sqrt{\wp - e_3}}\sqrt{e_1 - e_3}$$

$$= \frac{4(e_1{}^2 - e_3{}^2)}{2\sqrt{\wp - e_3}\ \sqrt{e_1 - e_3}} - \sqrt{e_1 - e_3}\,\frac{2\wp'\zeta + 4(\wp^2 - e_3{}^2)}{2\sqrt{\wp - e_3}}$$

$$= 2(e_1 - e_3)\operatorname{sn} u + 2\sqrt{e_1 - e_3}\operatorname{sn} u\cdot\zeta - \frac{2(e_1 - e_3)}{\operatorname{sn} u}.$$

If in the denominator of $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$, that is, in

$$\Psi_m = \frac{\sigma_3(mv)}{\sigma(v)^{m^2}},$$

we put

$$\sigma_3(mv) = \sigma_3(v)^{m^2}\Psi_m = y,$$

then we have

$$Dy = \frac{1}{m^2}\,y'' + \left(e_3 + \frac{1}{12}\,g_2 m^2 v^2\right)y.$$

But

$$\frac{Dy}{y} = \frac{D\Psi_m}{\Psi_m} + m^2 D\log\sigma_3(v).$$

$$\frac{Dy}{y} = \frac{D\Psi_m}{\Psi_m} + m^2\left\{\frac{\sigma_3''(v)}{\sigma_3(v)} + \left(e_3 + \frac{1}{12}g_2 v^2\right)\right\},$$

or

$$y' = \Psi_m'\sigma_3(v)^{m^2} + m^2\sigma_3(v)^{m^2-1}\Psi_m\sigma'(v),$$

$$\therefore\ \frac{y'}{y} = \frac{\Psi_m'}{\Psi_m} + m^2\left(\frac{\sigma_3'(v)}{\sigma_3(v)}\right).$$

Consequently

$$\frac{y'}{y} = \frac{\Psi_m'}{\Psi_m} + 2m^2\cdot\frac{\sigma_3'(v)}{\sigma_3(v)}\cdot\frac{\Psi_m'}{\Psi_m} + m^2(m^2-1)\left(\frac{\sigma_3'(v)}{\sigma_3(v)}\right)^2 + m^2\,\frac{\sigma_3''(v)}{\sigma_3(v)}.$$

Hence

$$\frac{D\Psi_m}{\Psi_m} + m^2\left(\frac{\sigma_3''(v)}{\sigma_3(v)} + e_3 + \frac{1}{12}\,g_2 v^2\right)$$

$$= \frac{1}{m^2}\left[\frac{\Psi_m''}{\Psi_m} + 2m^2\cdot\frac{\sigma_3'(v)}{\sigma_3(v)}\cdot\frac{\Psi_m'}{\Psi_m} + m^2(m^2-1)\left(\frac{\sigma_3'(v)}{\sigma_3(v)}\right)^2 + m^2\,\frac{\sigma_3'(v)}{\sigma_3(v)}\right] + e_3 + \frac{1}{12}g_2 v^2.$$

On the one hand, we have, by a known property of $\sigma_3(v)$,

$$\left(\frac{\sigma_3'(v)}{\sigma_3(v)}\right)^2 - \frac{\sigma_3''(v)}{\sigma_3(v)} - e_3 = \frac{(e_3 - e_2)(e_3 - e_1)}{\wp(v) - e_3},$$

so that the above expression reduces to

$$D\Psi_m = \frac{1}{m^2}\Psi_m'' + 2\cdot\frac{\sigma_3'(v)}{\sigma_3(v)}\cdot\Psi_m' + (m^2-1)\frac{(e_1 - e_3)(e_2 - e_3)}{\wp(v) - e_3}\,\Psi_m.$$

Now since $\Psi_m = \frac{\sigma_3(mv)}{\sigma_3(v)^{m^2}}$, or the denominator of $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ is a function of k and $\operatorname{sn} u$, so it is a function of e_2, e_3 and $\operatorname{sn} u = x$. Therefore

$$D\Psi_m = \frac{\partial \Psi_m}{\partial e_2} De_2 + \frac{\partial \Psi_m}{\partial e_1} De_1 + \frac{\partial \Psi_m}{\partial x} Dx.$$

On the other hand, $e_1 + e_2 + e_3 = 0$, so that

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{e_2 - e_3}{-e_2 - 2e_3},$$

$$2k\frac{\partial k}{\partial e_2} = \frac{-3e_3}{(-e_2 - 2e_3)^2},$$

$$2k\frac{\partial k}{\partial e_3} = \frac{3e_2}{(-e_2 - 2e_3)^2}.$$

$$\therefore \quad \frac{\partial \Phi_m}{\partial e_2} De_2 + \frac{\partial \Phi_m}{\partial e_3} De_3 = \frac{1}{2k}\frac{\partial \Phi_m}{\partial k} 4k^2(e_2 - e_1) = 2k(e_2 - e_1)\frac{\partial \Phi_m}{\partial k}.$$

Again

$$\sigma_3'(v) = \frac{\wp'(x)}{2\wp(v) - e^3} + \zeta,$$

$$\wp(v) - e_3 = \frac{e_1 - e_3}{x^2},$$

$$\wp'(v) = \frac{-2(e_1 - e_2)^{\frac{3}{2}} x'}{x^3},$$

$$\Phi_m' = \frac{\partial \Phi_m}{\partial x} x' \sqrt{e_1 - e_3},$$

$$\Phi_m' = \frac{\partial \Phi_m}{\partial x} \cdot \sqrt{(1 - x)(1 - k^2x^2)}\sqrt{e_1 - e_3},$$

$$\Phi_m'' = \left(\frac{\partial^2 \Phi_m}{\partial x} x'^2 + \frac{\partial \Phi_m}{\partial x} x''\right)\left(e_1 - e_3\right).$$

Hence our differential equation becomes

$$2m^2(e_2 - e_1)\frac{\partial \Phi_m}{\partial k} k + 2m^2\frac{\partial \Phi_m}{\partial x}\left[(e_1 - e_3)x + \sqrt{e_1 - e_3}\, x'\zeta - \frac{e_1 - e_3}{x}\right]$$

$$= \left[\frac{\partial^2 \Phi_m}{\partial x^2} x'^2 + \frac{\partial \Phi_m}{\partial x} x''\right](e_1 - e_2) + 2m^2\left[\frac{\wp'(v)}{2(\wp(v) - e_3)} + \zeta\right]\frac{\partial \Phi_m}{\partial x} x'\sqrt{e_1 - e_3},$$

which, divided by $e_1 - e_3$, and $\wp'(v)$ and $\wp(v) - e_3$ being expressed in terms of x, and the substitutions

$$x' = 1 - (1 + k^2)x^2 + k^2x^4, \quad x'' = -(1 + k^2)x + 2k^2x^3$$

being made, affords the so-called Jacobi's differential equation:

$$\frac{\partial^2 \Phi_m}{\partial x^2}\Big[1-(1+k^2)x^2+k^2x^4\Big]+\frac{\partial \Phi_m}{\partial x}\Big[\{(2m^2-1)k^2-1\}x-2(m^2-1)k^2x^3\Big]$$
$$+\,2m^2k(1-k^2)\frac{\partial \Phi_m}{\partial k}+m^2(m^2-1)k^2x^2\Phi_m=0.$$

But

$$\Phi_m = D_0 + D_2x^2 + \ldots + D_{2r}x^{2r} + \ldots + D_{m^2-3}x^{m^2-3} + D_{m^2-1}x^{m^2-1},$$

which being substituted we have

$$\{2D_2 + 4\cdot 3D_4x^2 + 6\cdot 5D_6x^4 + \ldots\}\{1-(1+k^2)x^2+k^2x^4\}$$
$$+\Big\{2D_2x+4D_4x^3+6D_6x^5+\ldots\Big\}\Big\{\big((2m^2-1)k^2-1\big)x-2(m^2-1)k^2x^3\Big\}$$
$$+\,2m^2k^2(1-k^2)\Big\{\frac{\partial D_0}{\partial k}+\frac{\partial D_2}{\partial k}x^2+\frac{\partial D_4}{\partial k}x^4+\ldots\Big\}$$
$$+\,m^2(m^2-1)k^2x^2(D_0+D_2x^2+D_4x^4+\ldots)=0.$$

Here we put the coefficient of x^{2r} equal to zero and get the formula

$$(2r+1)(2r+2)D_{2r+2}+4r\{(m^2-r)k^2-r\}D_{2r}+2m^2k^2(1-k^2)\frac{\partial D_{2r}}{\partial k}$$
$$+\{(m^2-2r+1)(m^2-2r+2)\}k^2D_{2r-2}=0.$$

But $D_0=1$, as we have seen before, so that we find successively

$$D_2 = 0,$$
$$D_4 = -\frac{m^2(m^2-1)}{4!}2k^2,$$
$$D_6 = \quad\frac{m^2(m^2-1)(m^2-4)}{6!}8(k^2+k^4),$$
$$D_8 = -\frac{m^2(m^2-1)(m^2-4)}{8!}4\{8\ m^2-9)(k^2+k^4)+(17m^2-69)k^4\},$$

. . . .

From these results and also from the general formula above given we see that D_{2r} is of the $(2r-2)^{\text{th}}$ degree in k. If the common factor of D_{2r-2} and D_{2r} is denoted by C, D_{2r+2} will also have the same factor. Other factors are to be made out. Here we write

$$D_{2r-2}=C\,[{}_{2r-2}D_0+{}_{2r-2}D_2k^2+{}_{2r-2}D_4k^4+\ldots+{}_{2r-2}D_{2s}k^{2s}+\ldots$$
$$+\,{}_{2r-2}D_{2r-4}k^{2r-4}],$$
$$D_{2r}=C\,[{}_{2r}D_0+{}_{2r}D_2k^2+{}_{2r}D_4k^4+\ldots+{}_{2r}D_{2s}k^{2s}+\ldots$$
$$+\,{}_{2r}D_{2r-4}k^{2r-4}],$$
$$D_{2r+2}=C\,[{}_{2r+2}D_0+{}_{2r+2}D_2k^2+\ldots+{}_{2r+2}D_{2s}k^{2s}+\ldots+{}_{2r+2}D_{2r}k^{2r}].$$

These being substituted in the above formula we obtain

$$(2r+1)(2r+2)\{{}_{2r+2}D_0 + {}_{2r+2}D_2 k^2 + \ldots + {}_{2r+2}D_{2s}k^{2s} + \ldots + {}_{2r+2}D_{2r}k^{2r}\}$$
$$+ \ldots$$
$$+ \ldots = 0,$$

whence the comparison of coefficients of k^{2s} gives

$$(2r+1)(2r+2)\,{}_{2r+2}D_{2s} + 4\{(r-s+1)m^2 - r^2\}\,{}_{2r}D_{2s-2}$$
$$+ 4(sm^2 - r^2)\,{}_{2n}D_{2s} + (m^2-2r+1)(m^2-2r+2)\,{}_{2r-2}D_{2s-2} = 0.$$

From this the expression for ${}_{2r+2}D_{2s}$ can be found. Also, $2s$ being replaced by $2r-2s+2$, we obtain the formula for ${}_{2r+2}D_{2r-2s+2}$; namely,

$$(2r+1)(2r+2)\,{}_{2r+2}D_{2r-2s+2} + 4(sm^2 - r^2)\,{}_{2r}D_{2r-2s}$$
$$+ 4\{(r-s+1)m^2 - r^2\}\,{}_{2r}D_{2r-2s+2}$$
$$+ (m^2-2r+1)(m^2-2r+2)\,{}_{2r-2}D_{2r-2s} = 0.$$

If

$$\begin{aligned} {}_{2r}D_{2r-2s+2} &= {}_{2r}D_{2s-2}, \\ {}_{2r}D_{2r-2s} &= {}_{2r}D_{2s}, \\ {}_{2r-2}D_{2r-2s} &= {}_{2r-2}D_{2s-2}, \end{aligned}$$

then we shall have

$${}_{2r+2}D_{2s} = {}_{2r+2}D_{2r-2s+2}.$$

But we see these identities hold for D_6 and D_8, so that also correct for D_{10}, and generally true.

In the same way from the value of D_{m^2-1} those of D_{m^2-3}, $D_{m^2-5}, \ldots$ will be found.

§ 4. In § 2 we have found

$$A(x^2) = (-1)^{\frac{m-1}{2}} (\sqrt{kx})^{m^2-1} D\left(\frac{1}{k^2x^2}\right).$$

But, as has been found,

$$D(x^2) = D_{m^2-1}x^{m^2-1} + D_{m^2-3}x^{m^2-3} + \ldots + D_{2r}x^{2r} + \ldots + D_2x^2 + D_0.$$

Hence we have

$$A(x^2) = (-1)^{\frac{m-1}{2}} (\sqrt{kx})^{m^2-1}\left[D_{m^2-1}\left(\frac{1}{kx}\right)^{m^2-1} + D_{m^2-3}\left(\frac{1}{kx}\right)^{m^2-3} + \ldots\right]$$
$$= (-1)^{\frac{m-1}{2}} \frac{1}{\sqrt{k}^{m^2-1}}\left[D_{m^2-1} + D_{m^2-3}k^2x^2 + \ldots\right].$$

§ 5. To express by a determinant the value of $\Phi_m = \frac{\sigma_3(mv)}{\sigma_3(v)^{m^2}}$, which is the denominator in the formula for $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$, we have to express the numerator and denominator of this fraction in terms of $\sigma(mv)$ and $\sigma(v)$, or

$$\sigma_3(v) = \frac{e^{-\eta_3(v)}\sigma(v+\omega_3)}{\sigma(\omega_3)},$$

$$\sigma_3(mv) = \frac{\sigma\{m(v+\omega_3)\} - \eta_3\left\{m^2 v + \frac{(m-1)^2 - 2(m-1)}{2}\omega_3\right\}}{\sigma(\omega_3)} \times (-1)^{\frac{m-1}{2}}.$$

Hence writing

$$\Phi(v) = \frac{1}{B}\begin{vmatrix} \wp'(v) & \wp''(v) \dots \wp^{(m-1)}(v) \\ \wp''(v) & \wp'''(v) \dots \wp^{(m)}(v) \\ \dots & \dots \\ \wp^{(m-1)}(v) & \wp^{(m)}(v) \dots \wp^{(2m-3)}(v) \end{vmatrix},$$

where

$$B = [2!\ 3!\ 4! \dots (m-1)!]^2,$$

the denominator in the formula of $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ will be equal to

$$\frac{(-1)^{\frac{m-1}{2}}\sigma\{m(v+\omega_3)\}}{\sqrt[4]{(e_1-e_3)(e_2-e_3)}^{\,m^2-1}\sigma(v+\omega_3)^{m^2}} = (-1)^{\frac{m-1}{2}}\frac{1}{\sqrt[4]{(e_1-e_3)(e_2-e_3)}^{\,m^2-1}}\Phi(v+\omega_3),$$

that is, equal to

$$(-1)^{\frac{m-1}{2}}\frac{1}{\sqrt[4]{(e_1-e_3)(e_2-e_3)}^{\,m^2-1}B}$$

$$\times\begin{vmatrix} \wp'(v+\omega_3) & \wp''(v+\omega_3) \dots \wp^{(m-1)}(v+\omega_3) \\ \wp''(v+\omega_3) & \wp'''(v+\omega_3) \dots \wp^{(m)}(v+\omega_3) \\ \dots & \dots \\ \wp^{(m-1)}(v+\omega_3) & \wp^{(m)}(v+\omega_3) \dots \wp^{(2m-3)}(v+\omega_3) \end{vmatrix}.$$

But, as

$$\wp(v) - e^3 = \frac{e_1 - e_3}{\operatorname{sn}^2 u},$$

$$\wp'(v) = \frac{-2(e_1-e_3)^{\frac{3}{2}}\sqrt{(1-\operatorname{sn}^2 u)(1-k^2\operatorname{sn}^2 u)}}{\operatorname{sn}^2 u},$$

$$\operatorname{sn}(u+\omega') = \frac{1}{k\operatorname{sn} u},$$

we have

$$\wp'(v+\omega_3) = -2(e_1-e_3)^{\frac{3}{2}}k\sqrt{\operatorname{sn}^3 u(1-\operatorname{sn}^2 u)(1-k^2\operatorname{sn}^2 u)}.$$

If we put with Jacobi

$$\sqrt{\operatorname{sn}^2 u\,(1-\operatorname{sn}^2 u)\,(1-k^2\operatorname{sn}^2 u)} = \sqrt{x^2(1-x^2)(1-k^2x^2)} = A,$$

we get

$$\wp'(v+\omega_3) = -2(e_1-e_3)^{\frac{3}{2}}kA.$$

Writing $-2(e_1-e_3)^{\frac{3}{2}}k = \lambda$ for brevity's sake, we have

$$\wp'(v+\omega_3) = \lambda A,$$

and

$$\wp''(v+\omega_3) = \lambda\frac{\partial A}{\partial u}\sqrt{e_1-e_3},$$

$$\wp'''(v+\omega_3) = \lambda\frac{\partial^2 A}{\partial u^2}\sqrt{e_1-e_3},$$

$$\cdots\cdots\cdots\cdots$$

$$\wp^{(r)}(v+\omega_3) = \lambda\frac{\partial^{r-1}A}{\partial u^{r-1}}\sqrt{e_1-e_3}^{\,r-1},$$

$$\cdots\cdots\cdots\cdots$$

Substituting these values in the above determinant, we obtain

$$(-1)^{\frac{m-1}{2}}\frac{\lambda^{m-1}}{\sqrt[4]{(e_1-e_3)(e_2-e_3)}^{\,m^2-1}B}$$

$$\times\begin{vmatrix} A & \sqrt{e_1-e_3}\frac{\partial A}{\partial u} & \sqrt{e_1-e_3}^{\,2}\frac{\partial^2 A}{\partial u^2} & \cdots & \sqrt{e_1-e_3}^{\,m-2}\frac{\partial^{m-2}A}{\partial u^{m-2}} \\ \sqrt{e_1-e_3}\frac{\partial A}{\partial u} & \sqrt{e_1-e_3}^{\,2}\frac{\partial^2 A}{\partial u^2} & \sqrt{e_1-e_3}^{\,3}\frac{\partial^3 A}{\partial u^3} & \cdots & \sqrt{e_1-e_3}^{\,m-1}\frac{\partial^{m-1}A}{\partial u^{m-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sqrt{e_1-e_3}^{\,m-2}\frac{\partial^{m-2}A}{\partial u^{m-2}} & \sqrt{e_1-e_3}^{\,m-1}\frac{\partial^{m-1}A}{\partial u^{m-1}} & \cdots & \cdots & \sqrt{e_1-e_3}^{\,2m-4}\frac{\partial^{2m-4}A}{\partial u^{2m-4}} \end{vmatrix},$$

or the factors being taken out of the determinant and reductions being made,

$$=(-1)^{\frac{m-1}{2}}\frac{2^{m-1}}{k^{\frac{(m-1)^2}{2}}B}\begin{vmatrix} A & \frac{\partial A}{\partial u} & \frac{\partial^2 A}{\partial u^2} & \cdots & \frac{\partial^{m-2}A}{\partial u^{m-2}} \\ \frac{\partial A}{\partial u} & \frac{\partial^2 A}{\partial u^2} & \frac{\partial^3 A}{\partial u^3} & \cdots & \frac{\partial^{m-1}A}{\partial u^{m-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{m-2}A}{\partial u^{m-2}} & \frac{\partial^{m-1}A}{\partial u^{m-1}} & \frac{\partial^{m}A}{\partial u^{3}} & \cdots & \frac{\partial^{2m-4}A}{\partial u^{2m-4}} \end{vmatrix}.$$

In the special case $m=5$, the quantities $\frac{\partial A}{\partial u}, \frac{\partial^2 A}{\partial u^2}, \cdots, \frac{\partial^6 A}{\partial u^6}$ being expressible as functions of $\frac{\partial A}{\partial z}, \frac{\partial^2 A}{\partial z^2}, \cdots$, we can get the denominator for $\operatorname{sn} 5u$ as was given by Jacobi.

$\frac{\partial A}{\partial u}, \frac{\partial^2 A}{\partial z}, \cdots$, may be expressed as functions of z, namely,

$$\frac{\partial A}{\partial u} = \frac{\partial A}{\partial z} \cdot 2xx' = \frac{\partial A}{\partial z} \cdot 2A = \frac{1-2(1+k^2)z+3k^2z^2}{A} \cdot 2A$$
$$= 2\{1-2(1+k^2)z+3k^2z^2\},$$
$$\frac{\partial^2 A}{\partial u^2} = \frac{\partial}{\partial z} \cdot \frac{\partial A}{\partial u} \cdot 2A = 4\{1-(1+k^2)z+6k^2z^2\}2A,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

A is of the degree $\frac{3}{2}$, and $\frac{\partial A}{\partial u}, \frac{\partial^2 A}{\partial u^2}, \cdots$ are each $\frac{1}{2}$ degrees higher than the preceding, and $\frac{\partial^n A}{\partial u^n}$ is of the $\left(\frac{3}{2}+\frac{1}{2}r\right)^{\text{th}}$ degree. These quantities are rational for odd values of r and contain the factor A for even values of r. The determinant is therefore of the form

$$(-1)^{\frac{m-1}{2}} \frac{2^{m-1}}{k^{\frac{(m-1)^2}{2}} B} \begin{vmatrix} R_0A & R_1 & R_2A & R_3 & \ldots & R_{m-2} \\ R_1 & R_2A & R_3 & \ldots & \ldots & R_{m-1}A \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{m-2} & R_{m-1}A & R_m & \ldots & \ldots & R_{2m-4}A \end{vmatrix}.$$

This quantity is evidently of a rational expression. For an odd value of r, R_r is of the degree $\frac{r+3}{2}$ in z, so that

$$R_r = r_0 + r_1 z + r_2 z^2 + \ldots + r_{\frac{r+3}{2}} z^{\frac{r+3}{2}}.$$

The value of R_{r+1} follows at once from that of R_r, for

$$R_{r+1}A = \frac{\partial R_r}{\partial z} \cdot 2A,$$

that is,

$$R_{r+1} = 2\frac{\partial R_r}{\partial z} = 2\left\{r_1 + 2r_2 z + \ldots + \frac{r+3}{2} r_{\frac{r+3}{2}} z^{\frac{r+1}{2}}\right\}.$$

By putting

$$R_{r+2} = (r+2)_0 + (r+2)_1 z + \ldots + (r+2)_s z^s + \ldots + (r+2)_{\frac{r+5}{2}} z^{\frac{r+5}{2}},$$

we can calculate the coefficients from those of R_r, namely,

$$R_{r+2} = \frac{\partial (R_{r+1}A)}{\partial z} \cdot 2A$$
$$= \{2r_2 + 3\cdot 2r_3 z + 4\cdot 3r_4 z^2 + \ldots + s(s-1)r_s z^{s-2} + \ldots\}2^2A^2$$
$$+ 2\{r_1 + 2r_2 z + \ldots + sr_s z^{s-1} + \ldots\}\{1-2(1+k^2)z+3k^2z^2\}.$$

Comparing the coefficients of z_s we have

$$(r+2)_s = 2^2\{(s+1)sr_{s+1} - (1+k^2)s(s-1)r_s + k^2(s-1)(s-2)r_{s-1}\}$$
$$+ 2\{(s+1)r_{s+1} - 2(1+k^2)sr_s + 3k^2(s-1)r_{s-1}\}$$
$$= 2\{(s+1)(2s+1)r_{s+1} - 2(1+k^2)s^2r_s + k^2(s-1)(2s-1)r_{s-1}\}$$
$$= 2\{(s+1)(2s+1)r_{s+1} - (1+k^2)s\cdot 2s\cdot r_s + k^2(s-1)r_{s-1}\}.$$

But $R_0 = 1$, so that $R_1, R_2, \ldots$ can be found in succession.

§ 6. The denominator of $\frac{\operatorname{sn} mu}{\operatorname{sn} u}$ being already found, the enumeration of its numerator does not involve any difficulty. In consequence of the result obtained in § 2, the numerator is the same as the denominator, only that u should be changed into $u + \omega'$, or what is the same thing, v should be changed into $v + \omega_3$, the result being multiplied with the quantity $(-1)^{\frac{m-1}{2}} (\sqrt{k}\operatorname{sn} u)^{m^2-1}$. Accordingly it is equal to

$$\frac{(\sqrt{k}\operatorname{sn} u)^{m^2-1}}{\sqrt[4]{(e_1-e_3)(e_2-e_3)}^{\,m^2-1} B}$$

$$\times \begin{vmatrix} \wp'(v+2\omega_3) & \wp''(v+2\omega_3) & \ldots & \wp^{(m-1)}(v+2\omega_3) \\ \wp''(v+2\omega_3) & \wp'''(v+2\omega_3) & \ldots & \wp^{(m)}(v+2\omega_3) \\ \cdots & \cdots & \cdots & \cdots \\ \wp^{(m-1)}(v+2\omega_3) & \wp^{(m)}(v+2\omega_3) & \ldots & \wp^{(2m-3)}(v+2\omega_3) \end{vmatrix}$$

$$= \frac{(\sqrt{k}\operatorname{sn} u)^{m^2-1}}{\sqrt[4]{(e_1-e_3)(e_2-e_3)}^{\,m^2-1} B} \begin{vmatrix} \wp'(v) & \wp''(v) & \ldots & \wp^{(m-1)}(v) \\ \wp''(v) & \wp'''(v) & \ldots & \wp^{(m)}(v) \\ \cdots & \cdots & \cdots & \cdots \\ \wp^{(m-1)}(v) & \wp^{(m)}(v) & \ldots & \wp^{(2m-3)}(v) \end{vmatrix}.$$

But

$$\wp'(v) = \frac{-2\operatorname{sn}' u\,(e_1-e_3)^{\frac{3}{2}}}{\operatorname{sn}^2 u} = \frac{-2\,(e_1-e_3)^{\frac{3}{2}}}{z^2} \sqrt{\frac{(1-z)(1-k^2 z)}{z}},$$

and if we put with Jacobi

$$B = \sqrt{\frac{(1-z)(1-k^2 z)}{z}},$$

the denominator will find its expression in the form of a determinant that contains $\frac{\partial B}{\partial u}, \frac{\partial^2 B}{\partial u^2}, \ldots$ As the process however resembles to that we have employed above, we shall here stop short of a repetition of the same reasoning.

Or, after having found R_r in the numerator, we may, by § 2, substitute $\frac{1}{kz}$ for z and multiply with

$$(-1)^{\frac{m-1}{2}} (\sqrt{\mu x})^{m^2-1} = (-1)^{\frac{m-1}{2}} (\sqrt{k}\, z)^{m^2-1},$$

for the same purpose.

M. KABA, ON THE FUNCTION THAT SATISFIES THE RELATION $f(z+1) = zf(z)$.[1]

The gamma function is defined by Euler by the definite integral

$$\Gamma(x) = \int_0^\infty e^{-y} y^{x-1}\, dy,$$

and by Gauss with the infinite product

$$\Gamma(x) = \lim_{m=\infty} \frac{m!\, m^{x-1}}{x(x+1)(x+2)\ldots(x+m-1)},$$

from which the relation

$$\Gamma(x+1) = x\Gamma(x)$$

is deduced as one of its properties.

But the converse problem remains always a question. Will be the gamma function the sole one-valued function that satisfies the relation $f(z+1) = zf(z)$? Or if not, what condition or what conditions will be required in order to produce such a function?

We here give a result of our studies about this subject.

1. When $|z|$ is increased without limit in the relation $f(z+1) = zf(z)$, we obtain

$$f(z+1) = zf(z),$$
$$f(z+2) = (z+1)f(z+1)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence when z assumes a sufficiently large modulus, p say, we have

$$f(p) = (p-1)(p-2)\ldots f(a).$$

It follows therefore that a number n cannot be determined such that $p^{-n} f(p)$ should assume an assignable value when $|p|$ is increased without limit. We have therefore an essential singular point in $z = \infty$.

We have next to distinguish three cases according as $z = 0$ is a zero point, a pole or an ordinary point of the function $f(z)$, which we propose to consider in turn.

2. First to consider the case for which $z = 0$ is a pole. In this case the relation $f(z+1) = zf(z)$ easily reveals $-1, -2, -3, \ldots$ as periodical poles. If the function have neither pole nor zero beside these, then the function will be, by a theorem of Weierstrass, of the form

1) Journal of Phys. Sch. in Tokyo, Vol. 11, 1902.

$$f(z)=\frac{e^{h(z)}}{\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{m}\right)\cdots\right]e^{g(z)}},$$

where we put

$$e^{g(z)}=e^{-z\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}+\cdots\right)},$$

and where $h(z)$ is an integral function.

The function $f(z)$ may also be written in the following form

$$f(z)=\frac{m!\,e^{h(z)}(1+\zeta_m)}{z(z+1)(z+2)\dots(z+m)\,e^{g_1(z)}},$$

where

$$e^{g_1(z)}=e^{-z\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right)},$$

and ζ_m is a number that approaches to zero with the indefinite increase of m.

Now we give to $h(z)$ a particular form $A+Bz$, and determine B under the condition $f(z+1)=zf(z)$. B will then assume the form

$$B=\log(m+1)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right).$$

Hence finally we get

$$f(z)=\frac{m!\,e^{A+z\log(m+1)}(1+\zeta_m)}{z(z+1)(z+2)\dots(z+m)}.$$

If now we assume the value of $f(z)$ for $z=1$ to be unity, then ultimately $e^A=1$, and consequently

$$f(z)=\frac{m!\,(m+1)^z(1+\zeta_m)}{z(z+1)(z+2)\dots(z+m)},$$

which is nothing but the gamma function defined by Gauss.

When we multiply it by $e^{h(z)}$, we obtain the general function for the present case in the form

$$f(z)=e^{h(z)}\Gamma(z),$$

where the integral function $h(z)$ is so constituted as to underlie the relation $h(z+1)=h(z)$.

3. The case where $z=0$ is a zero point of $f(z)$.

The periodical zeros pertaining to $z=0$ are $1, 2, 3, \dots$ and $-1, -2, -3, \dots$ Suppose the function possesses neither zeros nor poles beside these points. By Weierstrass' theorem we have

$$f(z)=\left[z\left(1-\frac{z}{1}\right)\left(1-\frac{z}{2}\right)\cdots e^{g_1(z)}\right]\left[\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots e^{g_2(z)}\right]e^{h(z)},$$

where

$$e^{g_1(z)} = e^{z\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}+\cdots\right)},$$

$$e^{g_2(z)} = e^{-z\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}+\cdots\right)},$$

and where $e^{h(z)}$ is an auxiliary factor. But since

$$z\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots = \frac{1}{\pi}\sin \pi z,$$

we have

$$f(z) = \frac{\sin \pi z}{\pi} e^{h(z)}.$$

Consequently from the relation $f(z+1) = zf(z)$ follows

$$e^{h(z+1)} = -ze^{h(z)},$$

that is,

$$h(z+1) - h(z) = \log z + i\pi,$$

a result that is impossible, for the right-hand side will become logarithmically infinite for $z=0$, while in the other side the same does not happen.

There is therefore no function, in this case, of the kind that is sought for.

4. Assume $z=0$ to be an ordinary point of the function $f(z)$. The relation $f(z+1) = zf(z)$ makes manifest 1, 2, 3, ... as periodic zeros. If the function be supposed to possess no other zeros or poles, we have from Weierstrass' theorem

$$f(z) = \left[\left(1-\frac{z}{1}\right)\left(1-\frac{z}{2}\right)\cdots e^{g_1(z)}\right]e^{h(z)},$$

a formula which may be written also

$$f(z) = \left[(1-z)(2-z)\ldots(m-z)\frac{1}{m!}e^{g_1(z)}\right]e^{h(z)}(1+\xi_m),$$

where ξ_m will vanish with the indefinite increase of m, and where we write $g_1(z) = z\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m}\right)$.

Now in the gamma function

$$\Gamma(z') = \frac{m!\, m^{z'-1}(1+\xi_m)}{z'(z'+1)(z'+2)\ldots(z'+m-1)},$$

if we put $z' = 1-z$, then

$$(1-z)(2-z)\ldots(m-z) = \frac{m!\,(1+\xi_m)}{\Gamma(1-z)\, m^z},$$

and we obtain

$$f(z) = \frac{e^{z\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right)}e^{h(z)}(1+\xi_m)}{m^z\,\Gamma(1-z)}.$$

Applying the relations $f(z+1)=zf(z)$ and $\Gamma(z)=z\Gamma(z)$, and taking into notice that the difference between $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}$ and $\log(-m)$ tends to approach a constant value, say c, for an infinite increase of m, we get

$$h(z+1)=h(z)+\log(z)+i\pi+c,$$

of which the right-hand side becomes logarithmically infinite for $z=0$, while the other member is only finite for the same value — a formula that is altogether absurd.

We have therefore no function sought for in this case.

5. We have heretofore considered only those cases in which we have neither zeros nor poles other than those that pertain periodically.

Now assume $a_1, a_2, a_3, \ldots, a_r$ and points pertaining to them to be zeros, and $b_1, b_2, b_3, \ldots, b_s$ with their pertaining points to be poles. The function that possesses these zeros and poles will have by Weierstrass' theorem the form

$$\varphi(z)=\frac{\prod_{n=1}^{r}\sin\pi(a_n-z)}{\prod_{n=1}^{s}\sin\pi(b_n-\pi)}\cdot e^{h(z)}.$$

By multiplying this function to those that we have already considered, we get

$$(1)\qquad f(z)=\frac{\prod_{n=1}^{r}\sin\pi(a_n-\pi)}{\prod_{n=1}^{s}\sin\pi(b_n-z)}\Gamma(z)e^{h(z)},$$

$$(2)\qquad f(z)=\frac{\prod_{n=1}^{r}\sin\pi(a_n-z)}{\prod_{n=1}^{s}\sin\pi(b_n-z)}\cdot\frac{\sin\pi z}{\pi}\cdot e^{h(z)},$$

$$(3)\qquad f(z)=\frac{\prod_{n=1}^{r}\sin\pi(a_n-z)}{\prod_{n=1}^{s}\sin\pi(b_n-z)}\cdot\frac{e^{z\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right)}}{m^z\,\Gamma(1-z)}\cdot e^{h(z)}.$$

When we change z into $z+1$ in these formulae, those parts designated under the sign Π do not change their absolute values, while the remaining factors behave themselves exactly as we have seen them in the cases already considered. Hence (1) is the sole functions that satisfy the relation $f(z+1)=zf(z)$.

Here the function $h(z)$ is to be such that satisfies the relation $h(z+1)=h(z)$, when r and s are both even or both odd. In the contrary case it is to be

$$e^{h(z+1)}=e^{h(z)}=e^{h(z)+i\pi},$$

that is,
$$h(z+1)=h(z)+i\pi.$$

We arrive therefore at the following condition:

The functions that satisfy the relation $f(z+1)=zf(z)$ are restricted to those functions that are termed functions of the Γ family, that have their expression in the form

$$f(z)=\Gamma(z)e^{h(z)}\cdot\frac{\prod_{n=1}^{r}\sin\pi(a_n-z)}{\prod_{n=1}^{s}\sin\pi(b_n-z)},$$

where the function h is of the form such that $h(z+1)=h(z)$ or $h(z+1)=h(z)+i\pi$, according as r and s are both even or both odd, or as r and s are the one even while the other odd.

M. KABA, ON PSEUDO-BIPERIODIC FUNCTIONS.[1]

The pseudo-biperiodic functions that have been hitherto studied appear in our limited knowledge to be restricted to those cases for which we have

(1) $$F(z+\omega_1)=e^{\varphi_1(z)}F(z),\quad F(z+\omega_2)=e^{\varphi_2(z)}F(z),$$

where the functions $\varphi(z)$ are of special forms. In the present paper we propose to carry our study on the function that satisfies the relations

(2) $$F(z+\omega_1)=S_1(z)F(z),\quad F(z+\omega_2)=S_2(z)F(z),$$

where $F(z)$ is a meromorphic function.

§ 1. Before we go in a consideration of the pseudo-biperiodic function we shall first explain about the pseudo-mono-periodic function, that satisfies the relation

(3) $$f(z+\omega)=S(z)f(z),$$

where $S(z)$ is a rational function.

1) Journal of Phys. Sch. in Tokyo, Vol. 12, pp. 162—167, April, 1903.

We take $a_1, a_2, \ldots, a_k$ for the zero points and $b_1, b_2, \ldots, b_l$ for the poles of the function $S(z)$. $a + m\omega$ and $b + m\omega$ are then respectively zeros and poles of the function $f(z)$. We form the function $\Phi(z)$ that possesses these zeros and poles

$$\Phi(z) = \frac{\prod_{n=1}^{k}\prod_{m=1}^{\infty}\left\{\left(1-\frac{z-a_i}{m\omega}\right)e^{\frac{z-a_i}{m\omega}}\right\}}{\prod_{j=1}^{l}\prod_{m=1}^{\infty}\left\{\left(1-\frac{z-b_j}{m\omega}\right)e^{\frac{z-b_j}{m\omega}}\right\}}.$$

Change z into $z+\omega$, and notice that $\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m}-\log m$ tends to approach a finite limit for $m=\infty$. We then have

$$\Phi(z+\omega) = M\cdot\frac{\prod_{i=1}^{k}(z-a_i)}{\prod_{j=1}^{l}(z-b_j)}\cdot\Phi(z)$$
$$= M\,S(z)\,\Phi(z),$$

where M is a constant.

Take $\varphi(z)$ to be a function that possesses the zeros and poles of $f(z)$ that do not arise from those of $S(z)$, and we have to see under what condition the function $\Phi(z)\,\varphi(z)$ should satisfy (3). Here we have

$$\Phi(z+\omega)\,\varphi(z+\omega) = S(z)\,\Phi(z)\,\varphi(z),$$
$$\therefore\ \varphi(z+\omega) = N\varphi(z),$$

so that the general function that satisfies (3) will be of the form $\Phi(z)\,\varphi(z)\,e^{h(z)}$, where $h(z)$ denotes a function such that $h(z+\omega)=h(z)$.

When $S(z)$ is a function that has a finite number of zeros and poles for finite values of z and a singular point at infinity, we see that $\varphi(z)$ will satisfy $\varphi(z+\omega)=e^{g(z)}\varphi(z)$. (Here and hereafter we indicate with $g(z)$ and like expressions integral functions.)

If $S(z)$ be a function that possesses a period ω_3 different from ω, the zeros and poles incongruent with ω_3 being $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_l$ respectively, we shall be required to take in the place of the previous $\Phi(z)$ the following function

$$(4)\quad \Phi(z+\omega) = \frac{\prod_{i=1}^{k}\prod_{n=-\infty}^{\infty}\prod_{m=1}^{\infty}\left\{\left(1-\frac{z-a_i}{n\omega_3+m\omega}\right)e^{\frac{z-a_i}{n\omega_3+m\omega}+\frac{1}{2}\frac{(z-a_i)^2}{(n\omega_3+m\omega)^2}}\right\}}{\prod_{j=1}^{l}\prod_{n=-\infty}^{\infty}\prod_{m=1}^{\infty}\left\{\left(1-\frac{z-b_j}{n\omega_3+m\omega}\right)e^{\frac{z-b_j}{n\omega_3+m\omega}+\frac{1}{2}\frac{(z-b_j)^2}{(n\omega_3+m\omega)^2}}\right\}}\cdot\Phi(z)$$
$$= e^{\bar{g}(z)}\,S(z)\,\Phi(z).$$

If we take $\varphi(z)$ and examine its property as we have done before, we shall have

$$\varphi(z+\omega) = e^{\bar{h}(z)}\varphi(z).$$

The function that satisfies (3) has therefore in general the form $\Phi(z)\,\varphi(z)\,e^{h(z)}$, where $h(z+\omega) = h(z)$, so that it follows that the pseudo-monoperiodic function has the form

$$\varphi(z+\omega) = e^{h(z)}\varphi(z).$$

§ 2. Here we come to the pseudo-biperiodic function.

Writing $z+\omega_2$ and $z+\omega_1$ for z in the two relations in (2), and comparing the two expressions for $F(z+\omega_1+\omega_2)$ thus obtained, we have

$$S_1(z+\omega_1)\,S_2(z) = S_2(z+\omega_1)\,S_1(z),$$

whence

$$\frac{S_1(z+\omega_2)}{S_1(z)} = \frac{S_2(z+\omega_1)}{S_2(z)} = \lambda(z), \quad \text{say}$$

$$\therefore\ S_1(z+\omega_2) = \lambda(z)\,S_1(z), \quad S_2(z+\omega_1) = \lambda(z)\,S_2(z).$$

If we assume that $\lambda(z)$ is a singly periodic function with the period ω_3 and the zeros $a_1, a_2, \ldots, a_k$ and the poles $b_1, b_2, \ldots, b_l$, we have from § 1

$$S_1(z) = \Phi_1(z)\,\varphi_1(z)\,e^{h_1(z)}, \quad S_2(z) = \Phi_2(z)\,\varphi_2(z)\,e^{h_2(z)}.$$

Consequently the relations in (2) will assume the forms

$$(2') \qquad F(z+\omega_1) = \Phi_1(z)\,\varphi_1(z)\,e^{h_1(z)}F(z),$$

$$(2'') \qquad F(z+\omega_2) = \Phi_2(z)\,\varphi_2(z)\,e^{h_2(z)}F(z).$$

But there is no function $F(z)$ that satisfies these relations, for it would have the zeros $a+p\omega_3+n\omega_2+m\omega_1$ that come from those of $\Phi_1(z)$ and that have three periods against the hypothesis of being a meromorphic function.

When $\lambda(z)$ is a doubly periodic function, it is very evident, that we have no solution.

Hence if $\lambda(z)$ have the zeros $a_1, a_2, \ldots, a_k$ and the poles $b_1, b_2, \ldots b_l$ at finite points, then $\Phi_1(z)$ and $\Phi_2(z)$ in $(2')$ and $(2'')$ will have the forms of (4), where the indices of ω and φ are of course to be suffixed to.

The zeros of $F(z)$ are of the three species indicated below:

I.	$a+n\omega_2+m\omega_1$	arisen from Φ_1 and Φ_2;
II.	$\alpha+\nu\omega_2+m\omega_1$	„ „ φ_1,
	$\alpha'+\mu\omega_1+n\omega_2$	„ „ φ_2;
III.	$\gamma+\mu\omega_1+\nu\omega_2$	otherwise arisen.

The poles of $F(z)$ are

I. $b + n\omega_2 + m\omega_1$ arisen from Φ_1 and Φ_2;

II. $\begin{cases} \beta + \nu\omega_2 + m\omega_1 & \text{,, ,,} \quad \varphi_1, \\ \beta' + \mu\omega_1 + n\omega_2 & \text{,, ,,} \quad \varphi_2; \end{cases}$

III. $\delta + \mu\omega_1 + \nu\omega_2$ otherwise arisen.

Here m denotes an integer from 1 to ∞ and n an integer from $-\infty$ to $+\infty$.

The function that possesses the zeros and the poles in I. is of the form

$$\Psi(z) = \frac{\prod\limits_{i=1}^{k}\prod\limits_{n=-\infty}^{\infty}\prod\limits_{m=1}^{\infty}\left\{\left(1-\frac{z-a_i}{n\omega_2+m\omega_1}\right)e^{\frac{z-a_i}{n\omega_2+m\omega_1}+\frac{1}{2}\frac{(z-a_i)^2}{(n\omega_2+m\omega_1)^2}}\right\}}{\prod\limits_{j=1}^{l}\prod\limits_{n=-\infty}^{\infty}\prod\limits_{m=1}^{\infty}\left\{\left(1-\frac{z-b_j}{n\omega_2+m\omega_1}\right)e^{\frac{z-b_j}{n\omega_2+m\omega_1}+\frac{1}{2}\frac{(z-b_j)^2}{(n\omega_2+m\omega_1)^2}}\right\}}.$$

By putting $z+\omega_1$ for z and by the application of the same formulae as in § 1 we obtain a result similar to (4), namely,

$$\Psi(z+\omega_1) = e^{\overline{G}(z)}\Phi_1(z)\Psi(z).$$

And similarly

$$\Psi(z+\omega_2) = e^{\overline{G'}(z)}\Phi_2(z)\Psi(z).$$

The functions, that possess the zeros and poles that arise from those of $\varphi_1(z)$ and $\varphi_2(z)$ respectively, have the relations

$$\psi_1(z+\omega_1) = e^{g(z)}\,\psi_1(z)\,\varphi_1(z), \quad \psi_1(z+\omega_2) = e^{\overline{g}(z)}\,\psi_1(z);$$

$$\psi_2(z+\omega_2) = e^{g'(z)}\,\psi_1(z)\,\varphi_2(z), \quad \psi_2(z+\omega_2) = e^{\overline{g'}(z)}\,\psi_2(z).$$

Next we assume a function $f(z)$ that possesses the zeros and poles III., and we find the condition that (2′) and (2″) should be satisfied by $\Psi(z)\,\psi_1(z)\,\psi_2(z)\,f(z)$, and we get

$$f(z+\omega_1) = e^{H_1(z)}f(z), \quad f(z+\omega_2) = e^{H_1(z)}f(z);$$

and in general $\Psi(z)\,\psi_1(z)\,\psi_2(z)\,e^{h(z)}$ will satisfy (2′) and (2″), $h(z)$ being an integral function that has the property $h(z+\omega_1+\omega_2) = h(z)$.

We come therefore to the conclusion that *the pseudo-biperiodic functions are restricted to those forms that possess the form of* (1).

G. HOSOKAWA, THE SERIES $\sum_n 1/n^{2m}$ SUMMED BY THE EXPANSION OF $e^{a \arcsin x}$ [1]

We denote by P_m the sum of all possible products of m factors that we can form with the reciprocals of the squares of all integers, and by P'_m and Q_m the corresponding sums for even and odd integers respectively. We also set

$$\sigma_{2m} = \frac{1}{1^{2m}} + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \dots$$

In the formula

$$e^{a \operatorname{arc\,sin} x} = 1 + \sum_1^\infty \frac{a^2(a^2+2^2)(a^2+4^2)\dots(a^2+\overline{2n-2}^2)}{(2n)!} x^{2n}$$

$$+ ax + \sum_1^\infty \frac{a(a^2+1^2)(a^2+3^2)\dots(a^2+\overline{2n-1}^2)}{(2n+1)^2} x^{2n+1},$$

$$\left(-\frac{\pi}{2} \leq \operatorname{arc\,sin} x \leq \frac{\pi}{2}\right),$$

we make the substitution $x = \sin\theta$ and integrate between the limits 0 and $\frac{\pi}{2}$, when it results

$$\frac{1}{a} e^{\frac{\pi}{2}a} - \frac{1}{a} = \frac{\pi}{2}\left\{1 + \sum_1^\infty \frac{a^2(a^2+2^2)(a^2+4^2)\dots(a^2+\overline{2n-2}^2)}{2^2\cdot4^2\cdot6^2\cdots(2n)^2}\right\}$$

$$+ \left\{a + \sum_1^\infty \frac{a(a^2+1^2)(a^2+3^2)\dots(a^2+\overline{2n-1}^2)}{1^2\cdot3^2\cdot5^2\cdots(2n+1)^2}\right\},$$

or rearranged according to ascending powers of a,

$$\frac{\pi}{2} + \frac{\pi^2}{2^2}\frac{a}{2!} + \frac{\pi^3}{2^3}\frac{a^2}{3!} + \dots + \frac{\pi^{2n}}{2^{2n}}\frac{a^{2n-1}}{(2n)!} + \frac{\pi^{2n+1}}{2^{2n+1}}\frac{a^{2n}}{(2n+1)!} + \dots$$

$$= \frac{\pi}{2}(1 + a^2P_1' + a^4P_2' + \dots + a^{2n}P_n' + \dots)$$

$$+ (aQ_1 + a^3Q_2 + \dots + a^{2n}P_n' + \dots).$$

A comparison of like powers of a gives

$$\frac{\pi}{2}P_n' = \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}, \quad \text{or} \quad P_n' = \frac{\pi^{2n}}{2^{2n}(2n+1)!},$$

$$\therefore\ P_n = \frac{\pi^{2n}}{(2n+1)!}; \quad \text{and} \quad Q_n = \frac{\pi^{2n}}{2^{2n}(2n)!}.$$

1) Journal of Phys. Sch. in Tokyo, Vol. 13, pp. 343—344, September, 1904.

Between P_m and σ_{2m} there exists the relation

$$\sigma_{2m} - P_1\sigma_{2m-2} + P_2\sigma_{2m-4} - \ldots + (-1)^{m-1}P_{m-1}\sigma_2 + (-1)^m m P_m = 0$$

(see Burnside and Panton, *Theory of Equations*).

From this formula we can calculate the value of σ_{2m} by recurrection. Writing $\sigma_{2m} = \frac{2^{2m-1} B_m}{(2m\ !}\pi^{2m}$, and the P's being replaced by their values, we obtain

$$\frac{2^{2m-1}B_m}{(2m)!} - \frac{2^{2m-3}B_{m-1}}{(2m-2)!\,3!} + \frac{2^{2m-5}B_{m-2}}{(2m-4)!\,5!} - \ldots$$
$$+ (-1)^{m-1}\frac{2B_1}{2!\,(2m-1)!} + (-1)^m\frac{m}{(2m+1)!} = 0,$$

or

$$2^{2m-1}\,{}_{2m+1}C_{2m}B_m - 2^{2m-3}\,{}_{2m+1}C_{2m-2}B_{m-1} + 2^{2m-5}\,{}_{2m+1}C_{2m-4}B_{m-2}$$
$$- \ldots + (-1)^{m-1}2\,{}_{2m+1}C_2B_1 = (-1)^{m-1}m.$$

From this we see that $B_1, B_2, B_3, \ldots$ are Bernouilli's numbers. (Chrystal, *Algebra*, Vol. II, Exercise 12.)

T. TAKAGI, A SIMPLE EXAMPLE OF CONTINUOUS FUNCTION WITHOUT DERIVED FUNCTION.[1])

The independent variable t, which we suppose for simplicity's sake to lie between 0 and 1, being expressed in the scale of 2, we put

$$(1) \qquad t = \sum^n \frac{c_n}{2^n}, \quad \text{where} \quad c_n = 0 \text{ or } 1,$$

and

$$\tau_n = \frac{c_n}{2^n} + \frac{c_{n+1}}{2^{n+1}} + \frac{c_{n+2}}{2^{n+2}} + \ldots,$$

$$\tau_n' = \frac{1}{2^{n-1}} - \tau_n,$$

and we form a function of t

$$f(t) = \sum_{1,\infty}^n \gamma_n, \quad \text{where} \quad \gamma_n = \tau_n \text{ or } = \tau_n',$$

according as $c_n = 0$ or $= 1$. Thus we have

$$f(t) = \sum \frac{a_n}{2^n}, \quad a_n = \nu_n \text{ or } \pi_n \text{ as } c_n = 0 \text{ or } 1,$$

1) Journal of Phys. Sch. in Tokyo, Vol. 14, pp. 1—2, December, 1904.

where π_n and ν_n stand for the number of those among $a_1, a_2, \ldots, a_n$ that are equal to 0 and 1 respectively; so that $\pi_n + \nu_n = n$.

The function $f(t)$ is evidently one-valued and continuous in the interval $0 \leq t \leq 1$. When t is a rational number of the form $\frac{m}{2^n}$, t will have two different expressions of the form (1). But these two expressions of t may be easily proved to give the same value of $f(t)$.

In order to prove the function of our description does not possess a derived function, we consider the value of $\frac{\Delta f}{\Delta t}$ for special values of Δt.

1. $$c_n = 0, \quad \Delta t = \frac{1}{2^n};$$

$$\frac{\Delta f}{\Delta t} = \pi_n - \nu_n - 2^{n+1}\tau_{n+1}.$$

2. $$c_{n-1} = 0, \quad c_n = 0, \quad \Delta t = \frac{1}{2^n};$$

$$\frac{\Delta f}{\Delta t} = \pi_n - \nu_n.$$

The values of $\frac{\Delta f}{\Delta t}$ for $\Delta t = \frac{1}{2^n}$ and $\Delta t = \frac{1}{2^{n+1}}$ will differ by $1 - 2^{n+1}\tau_{n+2}$, if $c_n = 0$, $c_{n+1} = 0$; and they will differ by $2^{n+1}\tau_{n+2}$, if $c_n = 0$, $c_{n+1} = 1$.

It follows therefore that, if we restrict ourselves to the value of Δt that possesses the form of $\frac{1}{2^n}$, the increment of $\frac{\Delta f}{\Delta t}$ remains finite, however great the value of n might be taken.

Consequently the function before us has no derived function.

T. HAYASHI, ON FUNCTIONS THAT SATISFY AN ADDITION-THEOREM RELATION.[1]

1. The problem of finding all functions that satisfy the relation

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)\,f(y)}$$

was first studied by C. Hitomi in the *Journal of the Tokyo Physics School*, Vol 4, where its solution is resorted by the calculus. But the problem may be treated otherwise in a simple way.

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1) Journal of Phys. Sch. in Tokyo, Vol. 15, pp. 1—2 and 83—84, December, 1905 and February, 1906. Two papers separately published.

By writing $f(x) = \tan\varphi(x)$, the relation before us will become

$$\tan\varphi(x+y) = \frac{\tan\varphi(x) + \tan\varphi(y)}{1 - \tan\varphi(x)\tan\varphi(y)}$$
$$= \tan\{\varphi(x) + \varphi(y)\},$$

and in consequence we have only to find all the functions that stand in the relation

$$\varphi(x+y) = \varphi(x) + \varphi(y) + n\pi,$$

a result that is said to have been arrived at by Cauchy.

Now put $x = y$. Then we get

$$\varphi(2x) = 2\varphi(x) + n\pi.$$

In this functional equation make the substitution

$$\varphi(x) = x\psi(x) - n\pi,$$

when it will be reduced to

$$\psi(2x) = \psi(x),$$

so that the function $\psi(x)$ is a constant independent of x. Denoting this constant by a, we have

$$\varphi(x) = ax - n\pi;$$

and therefore

$$f(x) = \tan(ax - n\pi) = \tan ax.$$

2. A function, that satisfies the addition-theorem

$$(1) \qquad f(x+y) = F\{f(x), f(y)\},$$

evidently satisfies the functional equation

$$(2) \qquad f(2x) = F\{f(x), f(x)\}.$$

But a function that satisfies (2) does not necessarily follow to satisfy (1).

For the purpose of finding all the functions that will satisfy (1), we first prove the

Theorem. If $f(x)$ be a function that satisfies the functional equation

$$(3) \qquad f(2x) = Ff(x)$$

then any other function that satisfies the same equation has the form $f(ax)$, a being an arbitrary constant. Hereby we assume $f(x)$ to be a uniform function, which is algebraical, or which possesses a single or double periods.

Let $f\varphi(x)$ be a function that satisfies (3). Then

$$f\varphi(2x) = Ff\varphi(x) = f\{2\varphi(x)\}.$$

$$\therefore\ \varphi(2x) = 2\varphi(x) + p,$$

where

$$f(x+p) = f(x).$$

$$\therefore\ \varphi(x) = ax - p$$

as in last article.

Consequently

$$f\varphi(x) = f(ax-p) = f(ax).$$

Here we notice that the number 2 in the preceding theorem may be replaced by any other number.

As an example we take the equation

$$f(zx) = 2f(x)\sqrt{1-f(x)^2}.$$

All the functions, that satisfy this equation, will be seen by our theorem to be $\sin ax$.

Since (2) is of the same form as (3), if a solution of (2) be $f(x)$, all other solutions of (2) will be of the form of $f(ax)$ in virtue of the theorem just proved. And as any function that satisfies (1) also satisfies (2), we can suppose that the function $f(x)$ has been so chosen that it satisfies (1) and (2) at the same time. If $f(x)$ satisfies (1), $f(ax)$ is also a solution of (1). Hence all functions that satisfy (2), that is, $f(ax)$, satisfy (1) too.

The above reasoning furnishes a mode of proof to the theorem given in last article.

When the complexity and entanglement of reasoning will not be rejected as loathsome, so there are still several other ways that serve for the establishment of the theorem before us.

K. OGURA, ON THE ADDITION-THEOREM OF THE CIRCULAR FUNCTIONS.[1]

Put

$$\varphi(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots,\ \psi(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

then, as we know, $\varphi(0) = 0$, $\psi(0) = 1$, and

$$\frac{d}{dx}\varphi(x) = \psi(x),\quad \frac{d}{dx}\psi(x) = -\varphi(x).$$

1) Journal of Phys. Sch. in Tokyo, Vol. 15, pp. 591—592, November, 1906

Next consider functions of x_1 and x_2 such that

$$U(x_1, x_2) = \varphi(x_1)\psi(x_2) + \psi(x_1)\varphi(x_2), \quad W(x_1, x_2) = x_1 + x_2.$$
$$V(x_1, x_2) = \psi(x_1)\psi(x_2) - \varphi(x_1)\varphi(x_2),$$

Then, above formulae being taken into account, we have

$$\frac{\partial(U, W)}{\partial(x_1, x_2)} = 0, \quad \frac{\partial(V, W)}{\partial(x_1, x_2)} = 0,$$

whence we infer that U and V are functions of W alone thus

$$F(x_1+x_2)=\varphi(x_1)\psi(x_2)+\psi(x_1)\varphi(x_2), G(x_1+x_2)=\psi(x_1)\psi(x_2)-\varphi(x_1)\varphi(x_2).$$

Here put $x_2 = 0$, then $F(x_1) = \varphi(x_1)$ and $G(x_1) = \psi(x_1)$, so that F and G are the functions φ and ψ themselves.

Therefore

$$\varphi(x_1 + x_2) = \varphi(x_1)\psi(x_1) + \psi(x_1)\varphi(x_2),$$
$$\psi(x_1 + x_2) = \psi(x_1)\psi(x_2) - \varphi(x_1)\varphi(x_2),$$

which establishes the addition-relation of the functions with which we have started.

K. OGURA, ON SOME FUNCTIONAL EQUATIONS.

I.

In the *Journal of the Tokyo Physics School*, Vol. 16, pp. 82—84 and 124—125, K. Katō published an investigation *on the functions that satisfy* $F(z)^2 + F(\alpha - z)^2 = 1$, which being a harbinger to K. Ogura's study, we here first give K. Katō's result.

1. *To find a function that satisfies*

(A) $$F(z)^2 + F(\alpha - z)^2 = 1,$$

we write

(1) $$F(z) + iF(\alpha - z) = e^{i\varphi(z)}$$

and replace z by $\alpha - z$; we multiply together the two formulae thus obtained, when we get, the relation (A) being referred to,

(2) $$i = e^{i\varphi(z)+i\varphi(\alpha-z)}, \text{ or } \varphi(z) + \varphi(\alpha - z) = \frac{\pi}{2},$$

since $i = e^{i\frac{\pi}{2}}$.

Again from (A), by factorising the left-hand side and using (1), we deduce

$$F(z) - iF(\alpha - z) = e^{-i\varphi(z)},$$

which, with (1), gives

$$F(z) = \frac{e^{i\varphi(z)} + e^{-i\varphi(z)}}{2}. \tag{3}$$

Thus *the function* $F(z)$, *that satisfies* (A), *also satisfies* (2) *and* (3), *and conversely.*

For from (2) we have

$$e^{i\varphi(z)+i\varphi(\alpha-z)} = i, \text{ or } e^{i\varphi(z)} = i^{-i\varphi(\alpha-z)},$$

or, squared and transposed,

$$e^{2i\varphi(z)} + e^{-2i\varphi(\alpha-z)} = 0, \text{ and } \therefore e^{2i\varphi(\alpha-z)} + e^{-2i\varphi(z)} = 0,$$

by virtue of which the expression $F(z)^2 + F(\alpha - z)^2$ obtained from (3) reduces to unity.

The formulae (2) and (3) serve for the determination of our function. Thus $\varphi(z) = z$, $\alpha = \frac{\pi}{2}$ satisfy (2), and then from (3) $F(z)$ will be seen to become $\cos z$, and we see that $\cos z$ satisfies

$$\cos^2 z + \cos^2\left(\frac{\pi}{2} - z\right) = 1.$$

Again (2) is satisfied also by $\varphi(z) = \frac{\pi}{2} - z$, $\alpha = \frac{\pi}{2}$, and at the same time from (3) $F(z)$ becomes $= \sin z$, so that $\sin z$ satisfies $\sin^2 z + \sin^2\left(\frac{\pi}{2} - z\right) = 1.$

2. To solve the equation (A) in another way, put $F(z) = \cos\varphi(z)$, and then

$$F(\alpha - z) = \pm \sin\varphi(z), \text{ or } F(z) = \pm\sin\varphi(z - \alpha).$$

$$\therefore \cos\varphi(z) = \pm\sin\varphi(\alpha - z), \quad \therefore \pm\varphi(z) = \frac{\pi}{2} \pm \varphi(\alpha - z) + 2n\pi.$$

Hence if we find a function $\varphi(z)$ that satisfies $\varphi(z) \pm \varphi(\alpha - z) = \frac{\pi}{2} + 2n\pi$, the function evidently makes good (A).

II.

K. Ogura's paper appears in the same *Journal* of April, May and July 1907.

1. The functional equation studied by K. Katō is solved by Babbage in the form

$$F(z) = \sqrt{\{\tfrac{1}{2} - \varphi(z) + \varphi(\alpha - z)\}},$$

and T. Hayashi has obtained its solution in

$$F(z) = \sqrt{\frac{\varphi(z)}{\varphi(z) + \varphi(\alpha - z)}}.$$

Here we try a little extension from Katō's considerations.

2. $\theta(z)$ being any known function, we consider the following system of four functional equations

$$\theta\{F(\alpha \mp z)\} \pm \theta\{F(z)\} = \beta,$$

which by the substitution of $\theta\{F(z)\} = \varphi(z)$ reduces to

$$\text{(X)} \qquad \varphi(\alpha \mp z) \pm \varphi(z) = \beta.$$

K. Katō's equation is one of these for which $\theta(z) = z^2$.

3. *Lemma. If $f(z)$ be a mono-periodic function with period 2ω, then the values of z that satisfy $f(z) = f(\zeta)$ are*

$$(i) \qquad z = \zeta + 2m\omega, \text{ or } z = 2\omega - \zeta + 2m\omega,$$

when $f(z)$ is an even function of the second order;

$$(ii) \qquad z = \zeta + 2m\omega,$$

when $f(z)$ is an odd function of the first order;

$$(iii) \qquad z = \zeta + 2m\omega, \text{ or } z = \omega - \zeta + 2m\omega,$$

when $f(z)$ is an odd function of the second order.

And in the case when $f(z)$ is a bi-periodic function with periods 2ω and $2\omega'$,

$$(iv) \qquad z = \zeta + 2m\omega + 2n\omega',$$

when $f(z)$ is an even function of the second order and the second class;

$$(v) \qquad z = \zeta + 2m\omega + 2n\omega', \text{ or } z = w - \zeta + 2m\omega + 2n\omega',$$

when $f(z)$ is an odd function of the second order and the second class.

To prove (*iii*). It is $f(z + 2\omega) = f(z)$. Let it be assumed

$$\text{(A)} \qquad f(a\omega - z) = f(z),$$

then

$$0 \leq z < 2\omega, \quad 0 \leq a\omega - z < 2\omega,$$
$$\therefore 0 < \alpha \leq 2.$$

But (A) does not hold when $a = 2$, so that a may be assumed to lie between 0 and 2. Writing $2 - a = b$, b also lies between the same limits.

From (A)

$$f(-z - b\omega) = f(z) = -f(z + b\omega),$$

or $z + b\omega$ being substituted for z,

$$f(z + 2b\omega) = f(z), \quad \therefore 2b\omega = 2\omega, \quad \therefore b = 1, \text{ i. e., } a = 1.$$

And it is easy to prove the reverse operation. The function being of the second order, our proposition follows established.

4. In

$$F(\alpha \mp z) = f[F(z)],$$

let $f(z)$ be a known one-valued function, and let it be satisfied by a known function $F_1(z)$ when $\alpha = \alpha_1$, or

$$F_1(\alpha_1 \mp z) = f[F_1(z)].$$

In these make the substitutions $F(z) = F_1\{\varphi(z)\}$ and $z = \varphi(z)$ respectively, then

$$F_1\{\varphi(\alpha \mp z)\} = f[F_1\{\varphi(z)\}], \quad F_1\{\alpha_1 \mp \varphi(z)\} = f[F_1\{\varphi(z)\}],$$

whence follows

$$F_1[\varphi(\alpha \mp z)] = F_1[\alpha_1 \mp \varphi(z)].$$

Therefore we see that the forms of $F_1(z)$ should be as given below.

(*i*) $\varphi(\alpha \mp z) = \alpha_1 \mp \varphi(z) + 2m\omega$, or $2\omega - \alpha_1 \pm \varphi(z) + 2m\omega$;

(*ii*) $\varphi(\alpha \mp z) = \alpha_1 \mp \varphi(z) + 2m\omega$;

(*iii*) $\varphi(\alpha \mp z) = \alpha_1 \mp \varphi(z) + 2m\omega$, or $\omega - \alpha_1 \pm \varphi(z) + 2m\omega$;

(*iv*) $\varphi(\alpha \mp z) = \alpha_1 \mp \varphi(z) + 2m\omega + 2n\omega'$, or

$$2\omega - \alpha_1 \pm \varphi(z) + 2m\omega + 2n\omega';$$

(*v*) $\varphi(\alpha \mp z) = \alpha_1 \mp \varphi(z) + 2m\omega + 2n\omega'$, or

$$\omega - \alpha_1 \pm \varphi(z) + 2m\omega + 2n\omega'.$$

Example. $F(\alpha - z) + F(z) = 1$.

Here $F_1(z) = \operatorname{tg} z$, and $\varphi(\alpha - z) + \varphi(z) = \frac{\pi}{2} + m\pi$.

5. Let $f(z)$ be a ϱ-valued function and let it be

$$F(\alpha \mp z) = f_\lambda[F(z)], \quad F_1(\alpha \mp z) = f_\mu[F_1(z)], \quad \lambda, \mu = 1, 2, 3, \ldots, \varrho.$$

In these we make the substitutions $F(z) = F_1\{\varphi(z)\}$ and $z = \varphi(z)$ respectively, and we get

$$F_1[\varphi(\alpha \mp z)] = f_\lambda[F_1\{\varphi(z)\}]$$

and

$$F_1[\alpha_1 \mp \varphi(z)] = f_\mu[F_1\{\varphi(z)\}].$$

If now a function $\theta_{\lambda\mu}(z)$, such as $f_\lambda(z) = \theta_{\lambda\mu}[f_\mu(z)]$, could be found, then from above formulae we have

$$F_1[\varphi(\alpha \mp z)] = \theta_{\lambda\mu}[F_1\{\alpha_1 \mp \varphi(z)\}].$$

Assuming $F_1(z)$ to be a periodic function of the N^{th} order, if we can find the functions $\sigma_{\lambda\mu}^{(j)}(z)$, $j = 1, 2, \ldots, N$, such that

$$F_1[z + \sigma_{\lambda\mu}^{(j)}(z)] = \theta_{\lambda\mu}[F_1(z)],$$

then by last two formulae we have

$$F_1[\varphi(\alpha \mp z)] = F_1[\{\alpha_1 \mp \varphi(z)\} + \sigma_{\lambda\mu}^{(j)}\{\alpha_1 \mp \varphi(z)\}],$$

whose solution will reduce to that of (X), if either

$$\alpha_1 \mp \varphi(z) + \sigma_{\lambda\mu}^{(j)}\{\alpha_1 \mp \varphi(z)\} = k \mp \varphi(z) \quad (k = \text{const.}),$$

i. e. $$\sigma_{\lambda\mu}^{(j)}(z) = k - \alpha_1 = \text{const.},$$

or

$$\alpha_1 \mp \varphi(z) + \sigma_{\lambda\mu}^{(j)}\{\alpha_1 \mp \varphi(z)\} = k' \pm \varphi(z),$$

i. e. $$\sigma_{\lambda\mu}^{(j)}(z) = k' + \alpha_1 - 2\pi.$$

In these two cases we have respectively

$$F_1[\varphi(\alpha_1 \mp z)] = F_1[k \mp \varphi(z)], \text{ and } F_1[\varphi(\alpha \mp z)] = F_1[k' \pm \varphi(z)].$$

Hence in the case of (i)

$$\varphi(\alpha \mp z) = k \mp \varphi(z) + 2m\omega, \text{ or } 2\omega - k \pm \varphi(z) + 2m\omega;$$

or

$$\varphi(\alpha \mp z) = k' \pm \varphi(z) + 2m\omega, \text{ or } 2\omega - k' \mp \varphi(z) + 2m\omega,$$

and similarly for other cases.

Ex. 1. In Katō's equation, it is $F_1(z) = \cos z$, and we have

$$\varphi(\alpha \mp z) + \varphi(z) = \frac{\pi}{2} + p\pi, \text{ or } \varphi(\alpha \mp z) = p\pi - \frac{\pi}{2}.$$

Ex. 2. In

$$F(z)^2 + F(\alpha - z)^2 - k^2 F(z)^2 F(\alpha \mp z)^2 = 1, \quad 1 \leq k \leq 1,$$

we have $F_1(z) = \operatorname{sn} z$, and

$$\varphi(\alpha \mp z) + \varphi(z) = mK + 2nK'i, \text{ or } \varphi(\alpha \mp z) - \varphi(z) = m'K + 2nK'i.$$

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6. *To solve the functional equations* (X).

In the equation $\varphi(\alpha - z) + \varphi(z) = \beta$, we put with Babbage

$$\varphi(z) = \frac{\beta}{2} + A[\chi(z), \chi(\alpha - z)],$$

where $\chi(z)$ is any function, and $A(x, y)$ any function that only changes sign when x and y are interchanged; and making $A(x,y) = y - x$ we get Babbage's solution.

The following is T. Hayashi's way of solution.

Differentiating in regard to z and writing $z = z_1 + \frac{\alpha}{2}$, we get

$$\varphi'\left(\frac{\alpha}{2} + z_1\right) = \varphi'\left(\frac{\alpha}{2} - z_1\right),$$

so that $\varphi'\left(\frac{\alpha}{2} + z_1\right)$ is an even function of z_1, $\chi(z_1)$ say, and

$$\varphi'\left(\frac{\alpha}{2}+z_1\right)=\chi(z_1),\ \text{ or } \varphi'(z)=\chi\left(z-\frac{\alpha}{2}\right),$$

$$\therefore\ \varphi(z)=\int_p^z\chi\left(\xi-\frac{\alpha}{2}\right)d\xi,$$

where the arbitrary constant p is to be determined from

$$\int_p^{\alpha-z}\chi\left(\xi-\frac{\alpha}{2}\right)d\xi+\int_p^z\chi\left(\xi-\frac{\alpha}{2}\right)d\xi=\beta.$$

7. The equation $\varphi(\alpha-z)-\varphi(z)=\beta$ is impossible unless $\beta=0$, as will be seen by substituting z by $\alpha-z$. In this case we have its solution in

$$\varphi(z)=S[\chi(z),\ \chi(\alpha-z)],$$

where S denotes a symmetrical function.

Again writing $z=z_1+\frac{\alpha}{2}$, we find $\varphi\left(\frac{\alpha}{2}+z_1\right)$ is an even function of z_1, so that

$$\varphi\left(\frac{\alpha}{2}+z_1\right)=\chi(z_1),\ \text{ or } \varphi(z)=\chi\left(z-\frac{\alpha}{2}\right).$$

8. In the equation $\varphi(\alpha+z)+\varphi(z)=\beta$ we can proceed in a similar way as in article 6, and we find

$$\varphi(z)=\int_p^z\chi(\xi)\,d\xi,\quad \int_p^{\alpha+z}\chi(\xi)\,d\xi+\int_p^z\chi(\xi)\,d\xi=\beta,$$

where χ is any periodic function with the period 2α.

9. In $\varphi(\alpha+z)-\varphi(z)=\beta$, $\varphi(z)$ is obviously a pseudo-periodic function with the period α.

If now $h(z)$ denote a particular solution, and χ the same as in last article, we have

$$\varphi(z)=h(z)+S\{\chi(z),\ \chi(\alpha+z)\},$$

for example

$$\varphi(z)=\frac{\beta}{\alpha}z+c+\chi(z)+\chi(\alpha+z).$$

Or

$$\varphi(z)=h(z)+\psi(z),$$

where ψ is a function with period α.

Ex. 1. Unless $\beta=0$, $\varphi(z)$ cannot be an even function. For on differentiation we see that $\varphi'(z)$ is a periodic function with period α, $\psi(z)$ say. Hence

$$\varphi(z)=\int_p^z\psi(\zeta)\,d\zeta,\ \ \beta=\int_p^{\alpha+z}\psi(\zeta)\,d\zeta-\int_p^z\psi(\zeta)\,d\zeta=\int_z^{z+\alpha}\psi(\zeta)\,d\zeta,$$

or putting $\zeta=\xi+z$,

$$\beta=\int_0^\alpha\psi(\xi+z)\,d\xi=\int_0^\alpha\psi[\alpha-(\xi+z)]\,d\xi=\int_0^\alpha\psi[-(\xi+z)]\,d\xi.$$

If now φ be an even function, that is, ψ an odd function,

$$\beta = \int_0^\alpha \psi(\xi + z)\, d\xi = -\int_0^\alpha \psi(\xi + z)\, d\xi,$$

which is impossible except for $\beta = 0$.

Ex. 2. To find a function $\varphi(z)$, such that

$$\varphi(z + \pi) = \varphi(z) + b, \quad \varphi(k\pi) = bk, \quad \varphi\left(\frac{2k+1}{2}\pi\right) = \infty,$$

where b is any constant and k a whole number. Here

$$h(z) = \frac{b}{\pi} z, \quad \psi(z) = \varphi(z) - \frac{bz}{\pi}, \quad \alpha = \pi,$$

$$\psi(k\pi) = 0, \quad \psi\left(\frac{2k+1}{2}\pi\right) = \infty,$$

$$\therefore \ \psi(z) = e^{g(z)} \operatorname{tg} z,$$

where $g(z)$ is any integral transcendental function.

$$\therefore \ \varphi(z) = \frac{3z}{\pi} + e^{g(z)} \operatorname{tg} z.$$

Ex. 3. The curves $\varrho = f(\omega)$, for which $f(\omega + \alpha) = f(\omega) + b$, are not necessarily restricted to $\varrho = \frac{b}{\alpha}\omega + c$. Therefore G. Loria seems to have erred when he says in his *Spezielle algebraische und transzendente Kurven der Ebene*, p. 433, that *Verlängert man alle Radienvektoren einer Archimedischen Spirale um dieselbe Strecke, so erhält man eine zweite, der ursprünglichen gleiche Spirale. Es ist hervorzuheben, daß diese Eigenschaft für die Archimedische Spirale charakteristisch ist.*

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10. We have next to solve the functional equations

$$\text{(Y)} \qquad \varphi(\alpha \mp z) \pm \varphi(z) = z + \beta.$$

The substitution of z by $\alpha - z$ transforms

$$\varphi(\alpha - z) + \varphi(z) = z + \beta \text{ into } \varphi(z) + \varphi(\alpha - z) = \alpha - z + \beta,$$

and so there is no function that satisfies the equation.

11. In $\varphi(\alpha - z) - \varphi(z) = z + \beta$, the same substitution reveals that the value of β must be $= -\frac{\alpha}{2}$. In this case we have

$$\varphi(z) = \frac{\alpha}{2} - \frac{z}{2} + S[\chi(z),\ \chi(\alpha - z)].$$

12. The solution of $\varphi(\alpha + z) + \varphi(z) = z + \beta$ is

$$\varphi(z) = \frac{z - 2\alpha + \beta}{2} + A[\chi(z), \chi(\alpha + z)],$$

where A signifies the form of function as given before and χ a function with the period 2α.

13. If $h(z)$ be a particular solution of $\varphi(\alpha + z) - \varphi(z) = z + \beta$, then generally ($\chi$ being as in article 12)

$$\varphi(z) = h(z) + S[\chi(z), \chi(\alpha + z)].$$

For example, in $\theta(z + b) = e^{-(b+2z)}\theta(z)$, we write $\log\theta(z) = -\varphi(z)$, when we get

$$\varphi(b + z) - \varphi(z) = 2z + b, \quad \therefore \varphi(z) = \frac{z^2}{2} + S[\chi(z), \quad \chi(b + z)],$$

$$\theta(z) = e^{-\frac{z^2}{b} - S[\chi(z), \chi(b+z)]}.$$

14. The general solution of $\varphi(\alpha + z) - \varphi(z) = z + \beta$ will be easily seen to be

$$\varphi(z) = \frac{z^2}{2\alpha} + \frac{2\beta - \alpha}{2\alpha} z + S[\chi(z), \chi(z + \alpha)],$$

where χ is a function with the period 2α.

---

15. *To solve the functional equations*

$$\varphi(\alpha \mp z) \pm \varphi(z) = f(z),$$

where $f(z)$ is any given function.

The substitution, $z = \alpha - z$, transforms

$$\varphi(\alpha - z) + \varphi(z) = f(z) \text{ into } \varphi(z) + \varphi(\alpha - z) = f(\alpha - z).$$

Hence it must be necessarily $f(z) = f(\alpha - z)$.
Then

$$f(z) = S[z, \alpha - z], \quad \therefore \varphi(z) = \frac{\chi(z) \cdot S[z, \alpha - z]}{\chi(z) + \chi(\alpha - z)}.$$

16. The equation $\varphi(\alpha - z) - \varphi(z) = f(z)$ does not hold unless

$$f(\alpha - z) + f(z) = 0, \text{ or } f(z) = A(z, \alpha - z).$$

In this case we have the general solution in the form

$$\varphi(z) = \frac{\chi(z) \cdot A(\alpha - z, \alpha)}{\chi(z) + \chi(\alpha - z)}.$$

17. In $\varphi(\alpha+z)+\varphi(z)=f(z)$, let it be

$$f(z)=a_0 z^n + a_1 z^{n-1} + \ldots + a_n,$$

and assume for $\varphi(z)$

$$\varphi(z)=b_0 z^n + b_1 z^{n-1} + \ldots + b_n + A[\chi(z),\ \chi(z+\alpha)].$$

Then the b's are to be determined from the following system

$$\begin{aligned} a_0 &= 2b_0 \\ a_1 &= b_0\, {}_nC_1\alpha + 2b_1, \\ a_2 &= b_0\, {}_nC_2\alpha^2 + b_1\, {}_{n-1}C_1\alpha + 2b^2, \\ &\cdots\cdots\cdots\cdots\cdots \\ a_n &= b_0\alpha^n + b_1\alpha^{n-1} + \ldots + b_{n-1}\alpha + 2b_n. \end{aligned}$$

In case $f(z)$ is of the form

$$\varphi(z)=a_1\cos z + a_2\sin z,$$

we assume

$$\varphi(z)=b_1\cos z + b_2\sin z + A[\chi(z),\ \chi(z+\alpha)],$$

and b_1 and b_2 will be determined from

$$b_1\cos\alpha + b_2\sin\alpha = a_1 - b_1,\quad -b_1\cos\alpha + b_2\cos\alpha = a_2 - b_2.$$

18. In $\varphi(\alpha+z)-\varphi(z)=f(z)$, (1) if we put

$$f(z)=a_0 z^n + a_1 z^{n-1} + \ldots + a_n,$$

then

$$\varphi(z)=b_0 z^{n+1} + b_1 z^n + b_2 z^{n-1} + \ldots + b_1 z + S[\chi(z),\ \chi(z+\alpha)],$$

where the function χ has the period of 2α; and we have

$$\begin{aligned} a_0 &= b_0\, {}_{n+1}C_1\alpha, \\ a_1 &= b_0\, {}_{n+1}C_2\alpha^2 + b_2\, {}_nC_1 a, \\ a_2 &= b_0\, {}_{n+1}C_3\alpha^3 + b_1\, {}_nC_2\alpha^2 + b_2\, {}_nC_1\alpha, \\ &\cdots\cdots\cdots\cdots\cdots \\ a_n &= b_0\alpha^{n+1} + b_1\alpha^n + \ldots + b_{n-1}\alpha^2 + b_n\alpha. \end{aligned}$$

(2) If we put $f(z)=a_1\cos z + a_2\sin z$, then

$$\varphi(z)=b_1\cos z + b_2\sin z + S[\chi(z),\ \chi(z+\alpha)],$$

where

$$b_1\cos\alpha + b_2\sin\alpha = a_1 + b_1,\quad -b_1\sin\alpha + b_2\cos\alpha = a_2 + b_2.$$

(3) If

$$f(z) = \log c \cdot \frac{\left(\frac{z}{\alpha} - a_1\right)\left(\frac{z}{\alpha} - a_2\right)\cdots\left(\frac{z}{\alpha} - a_m\right)}{\left(\frac{z}{\alpha} - b_1\right)\left(\frac{z}{\alpha} - b_2\right)\cdots\left(\frac{z}{\alpha} - b_n\right)},$$

then

$$\varphi(z) = \log c^{\frac{z}{\alpha}} \cdot \frac{\Gamma\left(\frac{z}{\alpha} - a_1\right)\Gamma\left(\frac{z}{\alpha} - a_2\right)\cdots\Gamma\left(\frac{z}{\alpha} - a_m\right)}{\Gamma\left(\frac{z}{\alpha} - b_1\right)\Gamma\left(\frac{z}{\alpha} - b_2\right)\cdots\Gamma\left(\frac{z}{\alpha} - b_n\right)} + S[\chi(z),\ \chi(z+a)].$$

---

19. Denoting by θ a functional operation and writing

$$\theta^2 z = \theta\{\theta(z)\},\ \theta^3 z = \theta[\theta\{\theta(z)\}],\ \ldots,$$

we have next *to solve the functional equation*

$$\varphi(\theta z) + \varphi(z) = f(z) \qquad (\theta^{2p+1} z = z).$$

$\theta z,\ \theta^2 z, \ldots$ being successively substituted for z, we get, from the results,

$$\varphi(z) = \tfrac{1}{2}[f(z) - f(\theta z) - f(\theta^2 z) - \ldots - f(\theta^{2p-1} z) + f(\theta^{2p} z)].$$

(See Babbage, *Functional Equations.*)

Examples.

(1) $$\varphi(z)^n + \varphi\left(\frac{1+z}{1-3z}\right)^n = a^n.$$

(2) To find a curve $y = f(x)$, the product of whose two ordinates equidistant from a point $(a, 0)$ on the x-axis is always equal to a^2. (Babbage.)

(3) To solve

$$\frac{\varphi(z) + \varphi(\theta z)}{1 - \varphi(z)\cdot\varphi(\theta z)} = f(z),$$

where $f(z)$ is any function and $\theta^2 z = z$. (Laplace.)

(4) $$\varphi(\theta z) + \varphi(z) = f(z) \qquad (\theta^{2p} z = z)$$

does not hold unless

$$\sum_{n=1}^{n=p} f(\theta^{2n-2} z) = \sum_{n=1}^{n=p} (\theta^{2n-1} z).$$

(5) $$\varphi(\theta z) - \varphi(z) = f(z) \qquad (\theta^m z = z),$$

where m is any positive integer, does not hold unless

$$\sum_{n=0}^{n=m-1} f(\theta^n z) = 0.$$

20. *A further solution of*

$$\varphi(z+\alpha)=\varphi(z)+f(z),$$

where $f(z)$ is any given rational function, for which we write

$$f(z)=\sum_{k=0}^{k=l} a_k z^k+\sum_{\lambda=1}^{\lambda=n}\sum_{\mu=1}^{\mu=m_\lambda}\frac{b_{\lambda\mu}}{(z-z_\lambda)^\mu}.$$

If we can find such functions $\xi_0(z)$ and $\xi_{\lambda\mu}(z)$ that satisfy

$$\text{(A)}\qquad \xi_0(z+\alpha)=\sum_{k=0}^{k=l} a_k z^k+\xi_0(z),$$

$$\text{(B)}\qquad \xi_{\lambda\mu}(z+\alpha)=\frac{b_{\lambda\mu}}{(z-z_\lambda)^\mu}+\xi_{\lambda\mu}(z),$$

then we shall have a general solution for $\varphi(z)$ in the following expression

$$\varphi(z)=\xi_0(z)+\sum_{\lambda=1}^{\lambda=n}\sum_{\mu=1}^{\mu=m_\lambda}\xi_{\lambda\mu}(z)+S[\chi(z),\,\chi(z+\alpha)],$$

where S stands for any symmetrical function, and $\chi(z)$ any function that satisfies $\chi(z+2\alpha)=\xi(z)$.

But we have already solved (A), so that the solution of (B) will be considered in the following.

Writing

$$\eta_{\lambda\mu}\left(\frac{z}{\alpha}\right)=\frac{\alpha^\mu}{b_{\lambda\mu}}\cdot\xi_{\lambda\mu}(z+z_\lambda),$$

we have

$$\eta_{\lambda\mu}(z+1)=\frac{1}{z^\mu}+\eta_{\lambda\mu}(z),\quad\text{or}\quad \eta_\mu(z+1)=\frac{1}{z^\mu}+\eta_\mu(z),$$

where we write $\eta_{\lambda\mu}(z)=\eta_\mu(z)$ for simplicity's sake.

(1) A particular solution of (B), when $\mu>1$, is

$$\eta_\mu(z)=-\sum_{n=0}^{n=\infty}\frac{1}{(z+n)^\mu},$$

whose right-hand side converges without condition, except for $z=1,2,\ldots$ We can also write

$$\eta_\mu(z)=(-1)^{\mu-1}(\mu-1)!\,\frac{d^\mu\log\Gamma(z)}{dz^\mu}.$$

(2) For $\mu=1$, the series

$$\sum_{n=0}^{n=\infty}\frac{1}{z+n}$$

is divergent, so that in this case we have to consider an infinite series

$$\eta_1(z) = -\varrho + \sum_{n=1}^{n=\infty}\left(\frac{1}{n} - \frac{1}{z+n-1}\right),$$

which converges except in $z = -1, -2, \ldots$; ϱ being Euler's or Mascheroni's constant. Or being rewritten we get

$$\eta_1(z) = \frac{d \log \Gamma(z)}{dz},$$

as a solution of our functional equation.[1])

21. In the *Journal of Mathematical-Physical Society in Tokyo* of 1901, T. Hayashi gives a study on the function

$$f(z+\omega_1) = e^{\Pi_1(z)} f(z), \quad f(z+\omega_2) = e^{\Pi_2(z)} f(z),$$

where

$$\Pi_1(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n,$$
$$\Pi_2(z) = b_0 z^n + b_1 z^{n-1} + \ldots + b_n;$$

and in the case $n=2$, arrives at the result that Π_1 and Π_2 must be of the same degrees in order that there should exist the function $f(z)$.

In the following it is intended to extend the proof to a general case.

First as the conditions that it should be

$$f[(z+\omega_2)+\omega_1] = e^{\Pi_1(z+\omega_2)} f(z+\omega_2) = e^{\Pi_1(z+\omega_2)+\Pi_2(z)} f(z),$$

and

$$f[(z+\omega_1)+\omega_2] = e^{\Pi_2(z+\omega_1)+\Pi_1(z)} f(z),$$

that is, that we should have

$$\Pi_1(z+\omega_2) + \Pi_2(z) = \Pi_2(z+\omega_1) + \Pi_1(z) + 2m\pi i,$$

we have the n equations

(1) $a_0\omega_2 - b_0\omega_1 = 0,$

(2) $\left(\frac{n}{1\cdot 2} a_0 \omega_2{}^2 + a_1\omega_2\right) - \left(\frac{n}{1\cdot 2} b_0\omega_1{}^2 + b_1\omega_1\right) = 0,$

. . . .

(n) $(a_0\omega_2^n + a_1\omega_2^{n-1} + \ldots + a_{n-1}\omega_2)$
$\qquad - (b_0\omega_1^n + b_1\omega_1^{n-1} + \ldots + b_{n-1}\omega_1) = 2m\pi i.$

Then, the irreducible zeros and poles of $f(z)$ being $x_1, x_2, \ldots x_\mu$ and $\beta_1, \beta_2, \ldots \beta_\nu$, respectively, we have, by Cauchy's theorem,

$$\int d \log f(z) = 2\pi i(\mu - \nu),$$

and

1) Consult O. Rausenberger, Lehrbuch der Theorie der periodischen Funktionen einer Variabeln.

$$\int z\, d \log f(z) = 2\pi i \left(\sum_1^{\mu} \alpha - \sum_1^{\nu} \beta \right),$$

where the integrations are to be carried out along the perimeter of the period-parallelogram $(p,\ p+\omega_1,\ p+\omega_2,\ p+\omega_1+\omega_2)$, where p is to be so selected that neither pole nor zero should be found on the sides.

From the results of these integrals, and (1) being taken into account, we have

$$(n') \quad (a_0 \omega_2^n + a_1 \omega_2^{n-1} + \dots + a_{n-1}\omega_2) - (b_0 \omega_1^n + b_1 \omega_1^{n-1} + \dots + b_{n-1}\omega_1) = 2\pi i (\mu - \nu)$$

and

$$(n+1) \qquad \psi_{n+1}(\omega_2) - \psi_{n+1}(\omega_1) = 2\pi i \left(\sum_1^{\mu} \alpha - \sum_1^{\nu} \beta \right),$$

where φ_{n+1} and ψ_{n+1} are rational functions of the degree $n+1$.

Since (n) and (n') are ultimately identical, and consequently $m = \mu - \nu$, we deduce the following results:

I. *Between the* $2(n+1)$ *constants* $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n$, *there exist* $(n+1)$ *relations.*

For $n = 1$, since

$$2\pi i (\mu - \nu) = a_0 \omega_2 - b_0 \omega_1,$$

a_0 and b_0 do not necessarily vanish together. But for $n > 1$, since $a_0 \omega_2 - b_0 \omega_1 = 0$, a_0 and b_0 must vanish together.

II. *The elliptic function of the third class being excepted,* $\Pi_1(z)$ *and* $\Pi_2(z)$ *are always of the same degree.*

By virtue of (1), we see, as a most notable sequence, that

III. (n') *and* $(n+1)$ *are altogether independent of* p.

When each of these conditions is fulfilled, by appropriately chosing $A_0, A_1, A_2, \dots, A_n$, we have the following solution

$$f(z) = e^{A_0 z^{n+1} + A_1 z^n + \dots + A_n z} \cdot \frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2) \dots \sigma(z-\alpha_\mu)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2) \dots \sigma(z-\beta_\nu)}.$$

SOME GEOMETRICAL THEOREMS IN THE JOURNAL OF THE MATHEMATICAL SOCIETY IN TOKYO.

I. *Given the vertical angle, the altitude and the sum or difference of the two sides of a triangle, to construct it.*

Y. Miyata's solution of this problem is given in *Vol. 2, Part 2, pp. 11—12* and *Vol. 4, pp. 320—321*, 1889 and 1891.

In the place it is noticed by the editor of the *Journal* that the problem is given in Wright's *Plane Geometry*, Vol. 3, and that it is solved in Chamber's *Geometry* which is however tedious, while the following solution is quite easy.

1. *The case of the sum of two sides given.*

In the triangle ABC draw the altitude AD, produce BA to E making $AE = AC$, form the rectangle $ADCF$ and produce CF to G making $CF = FG$. Let EG produced meet BC in F.

Then the points E, G, C lie evidently on a circle with A to centre. Hence follows easily

$$< CGH = \tfrac{1}{2} < EAC, \quad \therefore < CHG = AEC + \tfrac{1}{2} A.$$

Thus in the right-angled triangle CGH, the side CG and the angle CHG being known, CH can be constructed.

The two triangles EBC, FBH have equal angles at E and H, so that

$$BH : BE = BE : BC, \quad \therefore BH \cdot BC = BE^2.$$

Therefore BC can be constructed from the product and difference of BH and BC.

2. *The case of the difference of two sides given.*

On the side BC, that is greater than AC, take $AE = AC$. Take a point G as in the previous case and let GE, BC meet in H.

Then in the right-angled triangle CGH, the side CG and the angle G being known, CH can be known.

From the similarity of the triangles CEB, BEH, we deduce $CB \cdot BH = BE^2$, whence CB can be constructed, and hence our required construction follows.

3. *A second proof to the first case.*

In the triangle ABC circumscribe a circle and draw a diameter at right angles to the side AB cutting it in H. From D let fall perpendiculars DE, DF on BC and AC respectively. Of these the one will be within the quadrilateral $ADBC$ and the other without. Then

$$CE = CF, \quad DE = DF, \quad \triangle ADF = \triangle DBE, \quad AF = BE.$$

$$\therefore CE = CF = \tfrac{1}{2}(AC + BC).$$

In consequence the quadrilateral $DECF$ can be constructed. And drawing $CP \perp AB$, we have

$$ADBC = DECF = (CP + DH)\, BH.$$

Again since DBH is equal to half the angle ACB, the angles of the triangle DBH are known. Hence the ratio $BH : DH$ is a known quantity. Therefore $(CP + DH)\, DH$ is known.

Thus if we draw $CK \perp DG$, then the length of DK can be known from its difference with DH and the rectangle of these two lengths.

Therefore the point K can be constructed by the intersection of the semicircle on CD as diameter and the circle with D as centre and DK as diameter.

The point K being once found, the construction of the triangle ABC follows at once.

4. *Another proof to article 2.*

Since CG is the external bisector of the angle ACB, if we draw $GL \perp BC$ and $GM \perp AC$, the triangles CLG, CMG and GBL, are respectively congruent. Hence AM, BL are equal. Hence

$$CM = CL = \tfrac{1}{2}(AC - BC),$$

so that $CLGM$ can be constructed.

Again

$$2\triangle\, CMG = \triangle\, ACG - BCG = AGB - ACB = (GH - CP)\,BH,$$

so that the last member is a known quantity.

Since the angle BGH is half the angle ACB, the angles of the triangle BGH are known. Therefore $BH : GH$ is a known ratio.

Thus GK can be made out in length. Therefore the intersection of the circles on CG as diameter and with G as centre and GK as diametre determines the position of K, and the desired construction can be derived now in an easy manner.

II. T. Endō gives, in *Vol. IV, pp. 391—394*, 1891, the theorem: *The area of a triangle is equal to half the rectangle contained by a projection of a side in its plane, and the part of the line drawn to the side from the opposite vertex at right angles to the projection.*

This theorem is given as an extended form of one found in the *Sampō Shinsho* of 1830.

To prove this proposition, let $A'C'$ be a projection of AC, and draw $BD \perp A'C'$, where D is the point in which it meets the side AC.

We also draw $AC'' \parallel A'C'$, cutting CC' in C'', and $BE \perp AC$.

Since the triangles ACC'' and BDE are similar, we have

$$AC'' \text{ (or } A'C') : AC = BE : BD,$$

$$\therefore\ AC \cdot BE = A'C' \cdot BD,$$

that is,

$$\triangle\, ABC = \tfrac{1}{2} A'C' \cdot BD.$$

III. *The nine-point circle of a triangle touches the inscribed and escribed circles.*

The following proof to this variously proved theorem is one not difficult of understanding. (S. Shitō, *Vol. V, pp. 99—101*, 1892.)

In a triangle ABC, let the altitudes AG, BH meet in O, and let D be its circumcentre. Then the middle point N of OD is the centre of the nine-point circle.

Denoting the incentre of the triangle by I, draw IL, $DM \perp AG$, and $DF \perp BC$. Join the point I to O, D, N, A and join DA.

If then R and r signify the radii of the circumscribed and inscribed circles, the following theorems can be established.

1. $$IO^2 = 2r^2 - AC \cdot OG.$$
2. $$DI^2 = R^2 - 2Rr.$$
3. $$IO^2 = R^2 - 2AO \cdot OG.$$
4. $$IN = \frac{R}{2} - r.$$

Now I is the incentre and N the centre of the nine-point circle, r and $\frac{1}{2}R$ the radii of these circles. Hence the centre-line of the two circles is equal to the difference of their radii. Thus the two circles touch each other.

Similarly we can shew that the nine-point circle touches the escribed circles.

IV. *C. Hitomi, A proof to Pythagoras' theorem. (Vol. V, pp. 251 and 252.)*

Let ABC be a triangle right-angled at A. Describe squares on the three sides outside of the triangle.

Draw from A a parallel to BC and produce the side at B of the square on BC (AB is supposed to be greater than AC), thus dividing the square on AB into two triangles and two quadrilaterals.

Next from A let fall on BC the perpendicular AP, and from P draw parallels to AB and AC. On these lines complete a square which is equal to the square on AC, and produce one of its sides to meet the side opposite BC in the square on it. Thus the square on BC is decomposed into two triangles and two quadrilaterals besides the square equal to AC^2. These are easily seen to be congruent to those into which AB^2 is divided.

Therefore the Pythagorean theorem follows.

V. *To construct the perpendicular distance between two given parallels that subtends the greatest angle at a given point.* (This proof and the following are due to T. Ono. *Vol. V, pp. 297,* and *354—357, Vol. VI, pp. 12—14,* 1892.)

Through P, the given point, describe a circle with the centre on the perpendicular equally distant from two given parallels and let fall on them. Let AB and $A'B'$ be the perpendicular distances of these parallels cut off by the circle. These are the required lines.

For let A_1B_1 be any other perpendicular distance of the parallels, and draw $AE \| A_1P$, $BE \| B_1P$. The point E is on the parallel through P to the given parallels, hence outside of the circle. Hence

$$< APB > AEB \, (\therefore > A_1PB_1).$$

VI. *A quadrilateral with two pairs of opposite sides whose sums are equal can be circumscribed about a circle.*

To prove this theorem directly. (T. Ono.)

Describe a circle O to touch the three sides AB, AD and CD. Let P and Q be the points of contact with AB and CD respectively. (Here we suppose the point O is to that side of PQ on which AD lies.)

Then $BP + CQ = BC$, on account of the premise that the sums of opposite sides are equal. Hence if we take $BR = BP$, then also $CR = CQ$. Thus OPQ, BPR, CQR are isosceles triangles, on which our consideration rests.

We have then

$$PRQ = OPR + OQR,$$

for RPB and RQC being added to both sides, these become each two right angles.

$$OPR + RPQ + RQP = OPQ + RPQ + RPB,$$

for twice the left-hand side is equal to the sum of the angles of the triangle PQR, on account of above obtained relation, and the right member is evidently a right angle.

$$\therefore RQP = RPB = PRB,$$

whence we infer that AB and BC, and consequently CD on a similar ground, all touch the circle circumscribed about the triangle PQR.

Accordingly this circle identifies with the circle O and therefore the latter touches all the sides of the quadrilateral.

VII. A direct proof to the theorem: *A quadrilateral whose opposite angles are supplementary is inscribable in a circle.* (T. Ono.)

Describe the circumcircle O of the triangle ABD, and draw the tangents BE, DF. Then

$$ADC + ABC = 2 \text{ right angles} = OBE + ODF,$$

$$\therefore ODA + OBA = BAD = CDF + CBE.$$

But

$$BAD = BDF = CDF + CDB,$$

$$\therefore CDB = CBE,$$

which establishes our theorem at once.

VIII. *Of the quadrilaterals with sides of given lengths the inscribed quadrilateral has the greatest area.* (T. Ono.)

The four sides and the diagonals being denoted by a, b, c, d and m, n, we have for the area S

$$S = \tfrac{1}{4}\sqrt{\{(2mn + a^2 - b^2 + c^2 - d^2)(2mn - a^2 + b^2 - c^2 + d^2)\}}.$$

Hence S assumes a maximum when mn is so. But mn is $< ac + bd$, and in an inscribed quadrilateral the two quantities become equal.

Therefore the inscribed quadrilateral has the greatest area of all quadrilaterals whose sides are given in magnitudes.

If we write $a + b + c + d = 2s$, the expression for area, in case $mn = ac + bd$, becomes

$$S = \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}.$$

Consequently the maximum value does not change for any order in which the quadrilateral may be composed of its sides.

It may be proved also that the maximum polygon whose sides are given is one that can be inscribed in a circle, that the circle has the greatest area of all figures with a given perimeter, and that of the polygons whose sides are given all but one the polygon inscribed in the semicircle on the excepted side has the greatest area.

IX. *In a complete quadrilateral the orthocentres of the four triangles formed by its sides and the circumcentre of the triangle formed by the diagonals are all on a straight line perpendicular to the straight line joining the middle of the diagonals.* (T. Ono.)

Let L, M, N be the middle points of the diagonals of the complete quadrilateral $ABCDEF$, and PQR the triangle of the diagonals. Then $LP \cdot LQ = LC^2$, as is easy to see.

Hence the circle with AC as diameter intersects orthogonally with the circumcircle of the triangle PQR.

Similarly the circles described on BD and EF cut orthogonally the same circle.

The centres of these circles are L, M, N and are collinear; the circles form a coaxal system. Consequently the circumcentre of the triangle PQR is on the radical axis of the three circles.

Next let the three perpendiculars AX, FY, DZ of the triangle ADF meet in O_1. Then the circles L, M, N pass through X, Z, Y respectively, and

$$O_1A \cdot O_1X = O_1F \cdot O_1Y = O_1D \cdot O_1Y,$$

so that O_1 is on the radical axis of the circles.

The orthocentres of the triangles AEB, FBC, CDE may also be proved to lie on the radical axis of the three circles. And the radical axis is at right angles to the line LMN.

Therefore our proposition consists.

ON THE TRISECTION OF AN ANGLE.

1. The problem of trisecting an angle has attracted much attention in Japan as in other nations. And the Japanese scholars have come with some results, of which we here give a method derived by T. Yasutomi in the *Youth of Japan, Vol. 3, No. 9,* May, 1891. His method was only achieved after a long study of seven months. It seems to have been a discovery made some years before it was published.

Yasutomi's way is this:

With the vertex O of a given angle AOB as centre describe a circle cutting the sides in C and D. Draw a diameter of the circle at right angles to CD, meeting the circle in E on the opposite side of O as CD. Produce it to F so as to make EF one third of the diameter. Join F to G and H, the trisection points of CD, and produce the lines to meet the circle in K and L. Then KO and LO trisect the given angle.

This construction is given without proof as a matter of course. But the author is in a firm belief that his way should be strictly rigorous.

Hereupon K. Ishino disproves it in the *Mathematical Reports* of June, 1891, *pp. 9—12.* Thus writing $\angle OFK = \alpha$, $\angle OKF = \beta$, the given angle $= \theta$, we have

$$\cot \alpha = \frac{MF}{MG} = \frac{\cos\frac{\theta}{2} + \frac{5}{3}}{\frac{1}{3}\sin\frac{\theta}{2}} = 3\cot\frac{\theta}{2} + 5\operatorname{cosec}\frac{\theta}{2},$$

and

$$\sin\beta = \frac{OF}{OK}\cdot\sin\alpha = \frac{5}{3}\sin\alpha.$$

From these formulae, by assigning values to θ, we can calculate $\gamma = \angle KOL$, which is the constructed angle of trisection, and thus $\gamma - \frac{\theta}{3}$ are found for

$$340^0, \ldots, 200^0, \ldots, 90^0, \ldots, 50^0, 20^0, \ldots$$

to be

$$-87^0\,26', \quad -3^0\,36', \quad 16', \quad 4', \quad 0'$$

respectively.

Therefore this method can be applied only for small angles.

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2. In the *Mathematical Reports, Vol. 10, No. 98*, September, 1894, T. Yasutomi gives another way of approximate trisection of an angle.

Given the angle AOB, bisect it by OC. Draw $DOE \perp CO$, and describe a semicircle with the centre O cutting OE, OC, OF in E, C, F. Trisect the semicircle in m and n. On OD take OD' equal to its one third. Describe a semicircle with O as centre and with $\frac{1}{3}OD$ as radius, meeting OD, OA, OC, OE in D', l, F and E'. Bisect CF in O', and with O' as centre describe a semicircle $D''CE''$ equal to $D'FE'$, the diameters being parallel. Take $Cl' = Fl$. Join mD'' and produce it to meet the larger semicircle in P. Join PO. Then AOP is approximately one third of the given angle.

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3. In the *Oriental Journal of Science and Art, Vol. 8, pp. 518—520*, October, 1891, D. Kikuchi describes an instrument, invented by Y. Mitsuyoshi, by means of which the trisection of an angle can be effected.

Mitsuyoshi's instrument consists of three wooden arms AB, BC and ED, of which BC is joined to DE in a point on it, namely in C, where CB is made to revolve. The arm BA is connected at the end B to the end of CB, where it can revolve likewise. The other end A moves freely along ED. The instrument is provided with a hole at B, to fix it on paper. AB, BC, CD are made equal and CE should not be less than twice one of these equal lengths.

To trisect an angle by this instrument, we apply B at the vertex of the angle and fix BA along one of its sides. We then move ED,

keeping it always in contact with the point A, until D comes on another side produced. The angle BDA is one third of the given angle.

For, denote the angle to be trisected by ABF. Since the triangles ABC and BCD are both isosceles, we have

$$< CBD = CDB, \quad BAC = ACB, \quad \therefore BCD = 2 \cdot BDA,$$

$$\therefore < BAD + BDA = 3 \cdot BDA = ABF.$$

Here D. Kikuchi adds that it would be more convenient to fix the arm AB along the prolongation of a side in place of putting it along a side. In this case the point D is to be brought on the remaining side of the angle.

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4. C. Hitomi makes a review in the *Journal of the Society of Mathematics in Tokyo, Vol. 5, pp. 251—255* and *326—328*, 1892, of a trisection method by Benson in the British *Journal of Education* of 1888, which runs:

Let ASB be a given angle. Draw a circle O on the side AS as diameter. Erect SL, $PW \perp AS$, P being the middle of OA. From W, where PW cuts the circle, draw $WM \parallel PS$, meeting SL and SB in M and C. On WM produced take $WX = WC$. Draw XO cutting the circle in E and Y. Draw $ED \parallel SB$, meeting the circle again in D. Join DO and produce it to meet the circle in F. The points E and F being joined to S, the given angle will be trisected by these lines.

To this construction of Benson's, E. P. Matz, chief editor in the mathematical department of the Journal, makes a remark that it should be a complete solution obtained for the first time.

If however the above construction were correct, Y would be the middle of the arc SD, and conversely if Y be the middle of the arc SD, then the arcs AE, EF, FN will be all equal and the given angle actually trisected.

But this is not the case. In other words, the line ED, that joins E with the end D of the diameter drawn from the middle F of the arc EN, is not parallel to SB. This fact can be easily made out.

Benson's construction becomes exact only when the given angle is a right angle.

STUDIES ON PROBLEMS OF CONSTRUCTION GEOMETRICALLY IMPOSSIBLE.

I.

T. Hayashi, On the impossibility of a geometrical construction of a triangle whose three bisectors of angles are given in lengths.[1]

Analytical expressions that are capable to be geometrically constructed by means of a ruler and a compass are restricted to those that arise from a number of known quantities that are operated a finite number of times by rational operations and root-extractions of the second degree.

This theorem enables us in showing the impossibility of the problem of construction of a triangle, the lengths of whose three bisectors of angles are given; a problem considered at full length by Korselt in his paper, *Über das Problem der Winkelhalbierenden.*[2] This Korselt's demonstration is very elegant; but at the same time it lacks an easy understanding. We have therefore endeavoured to render the proof of this problem accessible to the domain of elementary mathematics, although the treatment should thereby loose much in generality and rigour.

The impossibility of this problem will be clearly made out when we show it is even so in a special case.

Let the three sides of a triangle be denoted by a, b, c, and the lengths of the bisectors of angles by α, β, γ. We shall have then for α^2, β^2, γ^2 the values

$$bc\left\{1-\left(\frac{a}{b+c}\right)^2\right\},\quad ca\left\{1-\left(\frac{b}{c+a}\right)^2\right\},\quad ab\left\{1-\left(\frac{c}{a+b}\right)^2\right\}.$$

Here we shall consider only the case of $\beta=\gamma$. In this case, also $b=c$; and we get

$$\alpha^2=b^2-\frac{a^2}{4},\quad \beta^2=ab\cdot\frac{a^2+2ab}{(a+b)^2}.$$

By eliminating a from these relations we have

$$\{8b^2(b^2-\alpha^2)-\beta^2(5b^2-4\alpha^2)\}^2=16b^2(b^2-\alpha^2)\{2(b^2-\alpha^2)-\beta^2\}^2,$$

or the substitutions $x=b^2$, $p=\alpha^2$, $q=\beta^2$ being made,

$$16(q-4p)x^3+(128p^2-16pq-9q^2)x^2+8p(3q^2-8p^2)x-16p^2q^2=0.$$

1) Journal of Phys. School in Tokyo, Vol. 8, pp. 221—223, July, 1899.

2) Zeitschrift für Mathematik und Physik, Bd. 42, 1897, und Zeitschrift für mathem. und naturwiss. Unterricht, Bd. 28, 1897.

If one root of this cubic equation were geometrically constructible, then its analytical expression would be of the form $f(p, q)$ or $f_1(p, q) + \sqrt{f_2(p, q)}$, where f, f_1, f_2 denote rational functions of p and q. In the latter case the cubic would have still another root

$$f_1(p, q) - \sqrt{f_2(p, q)},$$

so that it would be perfectly divisible by $\{x - f_1(p, q)\}^2 - f_2(p, q)$, leaving a quotient in the form of a linear function in x with rational coefficients. In both cases therefore the cubic could be divided perfectly by a linear function of x whose coefficients are rational in regard to p and q.

But it is evident that we can choose for p and q values that will render our cubic in an irreducible form. Such are, for example, $p = 1$ and $q = 16$, when the equation assumes the form

$$x^3 - 9x^2 + 23x - 16 = 0,$$

which is undoubtedly irreducible. No root of this equation can be constructed without taking the advantage of a root-extraction of the third degree.

Thus there is at least *a case* that is *impossible* to be constructed even when we conceive β and γ to be equal. *Our problem is therefore inconstructible in general.*

II.

On the extension of a problem of Pappus.

1. The problem of drawing a straight line through a given point that will be intercepted a given length between two given straight lines was solved by K. Tsuruta as an extension of Pappus' problem in the *Journal of the Mathematico-Physical Society in Tokyo, Vol. IV.*, and afterwards also by J. Midzuhara in a different way in the same volume of the same *Journal.*

These two methods are as given below:

2. K. Tsuruta runs thus:

A solution of this problem may be made to depend upon the following two theorems whose establishment is very easy.

(1) *The locus of the middle point of a straight line of given length and included between any two given straight lines is an ellipse.*

(2) *The locus of the centre of gravity of the triangle formed by a straight line through a given point with two given straight lines coplanar with the point is a hyperbola.*

The construction of our problem is then as follows:

Describe an ellipse (1), *taking the constant length* $= \frac{2}{3}$ *of the given length; then describe a hyperbola* (2), *the given point being taken as the fixed point.*

Next describe another ellipse similar to and concentric with the ellipse already described, the former bearing to the latter the ratio of similitude of $3:2$.

The minor ellipse will in general intersect with the hyperbola in four points.

Again draw from the intersection of the given straight lines the four radii vectores through these points and let them intersect the ellipse in P_1, P_2, P_3, P_4.

Then the straight lines OP_1, OP_2, OP_3, OP_4 *have their segments included between the given straight lines equal to the given length.*

This paper of K. Tsuruta's bears the date of May, 1889.

3. J. Midzuhara's construction will be reproduced in the following lines:

The solution of this problem may otherwise be made to depend upon the following two theorems which can easily be proved.

(1) The same as the theorem (1) in Tsuruta's paper.

(2) *The locus of the middle point of a straight line passing through a given point and included between any two given straight lines is a hyperbola.*

There is in general four different positions of the straight line satisfying the condition of the problem, and it is remarkable that the middle point of the given portion, that is to say, *the four points of intersection of ellipse* (1) *and hyperbola* (2) *lie on a fixed circle* This circle may be obtained directly from the following geometrical consideration.

Let AI and EI be the two given straight lines and P the given point. Draw PA and PB parallel to these lines. From the points A and B let fall perpendiculars AO and BO on the lines EI and AI, respectively; then the point of their intersection O will be the centre of the circle in question.

Let PQS represent a position of the required straight line and QS be equal to the given length. Join Q, I, S and the middle point M of QS with O. Draw QD and SF perpendicular to PA and IE respectively, and also PH at right angles to EI. We also take E for the point where OA meets the given line IE, and C for the point where QD intersects with AP. Draw OG perpendicular to AI.

Now

$$QO^2 = IQ^2 + IO^2 - 2 \cdot IQ \cdot IG,$$
$$SO^2 = SI^2 + IO^2 + 2 \cdot SI \cdot IE,$$

whense by addition we have

$$QO^2 + SO^2 = IQ^2 + SI^2 + 2 \cdot IO^2 + 2 \cdot SI \cdot IE - 2 \cdot IQ \cdot IG.$$

But

$$IQ^2 + SI^2 = SQ^2 - 2 \cdot SI \cdot ID,$$

(1) $$\therefore\ QO^2 + SO^2 = SQ^2 + 2 \cdot IO^2 + 2 \cdot SI \cdot IE - 2 \cdot SI \cdot ID - 2 \cdot IQ \cdot IG$$
$$= SQ^2 + 2 \cdot IO^2 + 2 \cdot SI \cdot DE - 2 \cdot IQ \cdot IG.$$

By similar triangles we get

$$\frac{IG}{FB} = \frac{IB}{SB} = \frac{PQ}{PS} = \frac{QC}{PH} = \frac{QC}{AE} = \frac{CA}{EI} = \frac{DE}{IE},$$

whence

$$\frac{IG}{FB} = \frac{DE}{IE} \text{ or } \frac{IG}{DE} = \frac{FB}{IM}.$$

Also

$$\frac{SI}{QI} = \frac{SB}{BP} = \frac{SF}{PH} = \frac{SF}{AE} = \frac{FB}{IE}.$$

From the last two formulae we have

$$\frac{IG}{DE} = \frac{SI}{QI},$$

or

$$SI \cdot DE - QI \cdot IG = 0.$$

This being substituted in (1),

$$SO^2 + SO^2 = SQ^2 + 2 \cdot IO^2.$$

But

$$QO^2 + SO^2 = 2 \cdot MO^2 + 2 \cdot \frac{SQ^2}{4}.$$

Hence

$$MO^2 = \frac{SQ^2}{4} + IO^2 = \text{const.}$$

From this it is evident that the middle point M lies on the circle whose radius = hypotenuse of a right-angled triangle having $\frac{1}{2} \cdot SQ$ and IO for its sides, the point O being the centre.

4. The above ways of solutions are both carried on the employment of the conics. It is still questionable whether the problem could be constructed by means of merely elementary geometrical requirements. The question was taken up by T. Hayashi, who arrived at the following result:[1])

1) Journal of Phys. Sch. in Tokyo, Vol. 10, pp. 1—4, December, 1900.

Let P be the given point and OA and OB the given straight lines, the angle included between them being ω.

Suppose for a moment our problem solved, and let PQR be a straight line satisfying the required condition. We denote thereby $QR = a$ for the given length. We shall have then

$$QR^2 = OQ^2 + OR^2 - 2 \cdot OQ \cdot OR \cos \omega,$$

or

$$a^2 = x^2 + y^2 - 2xy \cos \omega.$$

From this and the relation

$$(y + n) : m = y : x,$$

where m and n stand for the coordinates of the point P referred to the given lines as axes, we eliminate y and obtain the equation

$$x^4 - 2(m + n \cos \omega)x^3 + (m^2 + n^2 + 2mn \cos \omega - a^2)x^2 + 2a^2mx - a^2m^2 = 0,$$

no root of which is capable to be geometrically constructed, which we are going to point out.

For the demonstration of this subject it will be only sufficient if we take a special case and show the impossibility of a geometrical construction even in that case. For that purpose we choose the values $\omega = 90^0$ and $m = n = 2$; when our equation assumes the form

$$x^4 - 4x^3 + (8 - a^2)x^2 + 4a^2x - 4a^2 = 0,$$

or a^2 being replaced by 44,

(1) $$x^4 - 4x^3 - 36x^2 + 4 \times 44x - 4 \times 44 = 0.$$

Now the general equation of the fourth degree

(2) $$ax^2 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

will be transformed by the substitutions

$$Z = ax + b, \quad H = ac - b^2,$$
$$I = ae - 4bd + 3c^2, \quad G = a^2d - 3abc + 2b^3,$$

into

(3) $$Z^4 + 6HZ^2 + 4GZ + a^2I - 3H^2 = 0.$$

Whenever any root of (2) is constructible by the applications of rational operations and a number of root-extractions of the second degree, so it should be also the case with one root of (3), and vice versa. And if one root of (3) were constructible, the same equation should have necessarily another root that is also constructible.

If we put $Z = \sqrt{p} + \sqrt{q} + \sqrt{r}$, then p, q and r will be the roots of the cubic

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0.$$

Hence the problem of the constructibility will be decided by a study of this last equation.

By replacing the constants in the equation (2) by those in (1), the last equation will take the form

$$t^3 - 21t^2 + 87t - 144 = 0,$$

which is irreducible as will be easy to see. It is therefore the construction impossible.

Consequently it follows that our problem is geometrically insoluble.[1])

III.

T. Hayashi, On an extension of Castillon's problem.[2])

Among the problems of elementary geometry there is one called by the name of Castillon. It is this:

To inscribe in a given circle a triangle whose sides should pass througth three given points.

This problem was first proposed for solution in 1742 by Cramer to Castillon, who published his solution in 1776. In the year 1780 appeared its various solutions by Lagrange, Euler, Lhulier, Fuss and Lexell.

A Neapolitan youth of sixteen, Oltaiano by name, succeeded in so far as to extend this problem into another to inscribe a polygon of n sides, whose sides pass through n given points. Oltaiano's solution for his extended problem was very simple and yet so elegant and astonishing that he was met with a great surprise.

Still later Poncelet replaced the circle with any conic and put some restraints on the positions of the points where the sides are to be made to pass.

The problem we are going to solve in this place is one that may also be termed in a sense as an extension from the problem of

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1) From the irreducibility of an equation of the fourth degree it does not immediately follow that its roots are inconstructible; but the same property in the cubic imposes on its roots a necessary inconstructibility through the geometrical means.

2) Journal of Phys. Sch. in Tokyo, Vol. 13, pp. 204—208, May, 1904.

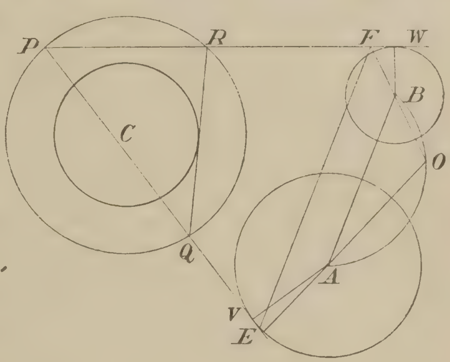
Castillon. Namely, we propose *to construct a triangle inscribed in a given circle, its three sides being required to touch three given circles;* this is a problem contained without solution on the closing page of *Lachlan's Treatise on Modern Pure Geometry.*

The solution of this problem being required, it was submitted by the author to the study of K. Katō, who came to the result that it is impossible for a mere geometrical construction.

The following is the answer given by K. Katō.

Let S be the circle wherein a triangle is to be inscribed, and let A, B, C be the three given circles, which the sides of the proposed triangle have to touch. We conceive for simplicity's sake the two circles S and C to be concentric; for it will suffice for our purpose to show the impossibility of our problem even in such a special case.

We assume PQR to be the inscribed triangle. The side QR, that touches C and is at the same time a chord of the concentric circle S, has a prefixed length. Hence the angle at P that stands on the circumference opposite this chord has also a prefixed magnitude, α say. The point P will be therefore an intersection of S with the locus of the point of intersection of two straight lines that touch the circles A and B respectively and contain a certain angle. If this locus should be found to be a curve of the third or higher degree, then our task must be deemed already done. For there is no curve other than the circle and the straight line, whose intersections with a circle can be determined by a geometrical construction. *(Petersen, Théorie des Équations algébriques.)*



Let the two sides under consideration touch with the circles A and B in the points V and W. Describe on AB as chord an arc that contains the angle equal to the suplement of α. In the segment thus constructed, inscribe an angle AOB whose sides AO and BO are in the ratio of the radii of the circles A and B. The position of O is determinate. If we produce OA and OB to meet PV in E and PW in F respectively, the triangles AEV and BFW are similar, so that AE, BF are in the ratio as AO and BO. Consequently AB and EF are parallel. Hence the angle $OPF = BAO = \text{const.} = \beta$ say.

We take O for the pole of polar coordinates and OBF for the initial line, and we find the equation to the locus of P in the form

$$\frac{r \sin \beta}{p} = 1 + \frac{a}{p} \sin (\theta + \beta),$$

where we put $a = OB$ and $p = BW$. This equation evidently represents the curve inverse to the conic

$$\frac{k \sin \beta}{rp} = 1 + \frac{a}{p} \sin (\theta + \beta),$$

one of the foci being taken for the centre of inversion. Such a curve is in general of the fourth degree and does not separate into two curves of lower degrees.

It follows therefore immediately the inconstructibility of the problem.

IV.

A problem considered by K. Katō.[1])

Given the orthocentre, the incentre and the circumcentre of a triangle to construct it.

Denoting the orthocentre by P, the incentre by I, the circumcentre by O and the centre of gravity by G, and also the radii of the inscribed and circumscribed circles being designated by r and R, the three points P, G, and O are collinear, and we have

$$PG = GO, \quad GI = \tfrac{1}{2} R - r,$$

and

$$OP^2 = qR^2 - (a^2 + b^2 + c^2), \quad OI^2 = R^2 - 2Rr,$$

as will be seen from *Lock, Higher Trigonometry, p. 134.*

It follows therefore that the positions of the orthocentre, incentre and circumcentre can be determined, if we know the magnitudes of R, r and $k^2 = a^2 + b^2 + c^2$. Hence we have only to consider the problem of constructing a triangle from the last data.

Here we have

$$R = \frac{abc}{4\triangle}, \quad r = \frac{\triangle}{s},$$

where

$$s = \tfrac{1}{2}(a + b + c), \quad \triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

1) Journal of Phys. Sch. in Tokyo, Vol. 14, pp. 123—126, March 1905.

From these relations it is easy to draw out the expressions

$$a + b + c = 2s = 2\sqrt{\frac{8Rr + 2r^2 + k^2}{2}},$$

$$ab + bc + ca = \frac{16Rr + 4r^2 + k^2}{2},$$

and

$$abc = 4Rr\sqrt{\frac{8Rr + 2r^2 + k^2}{2}};$$

so that a, b, c will form three roots of the cubic

$$x^3 - \sqrt{16Rr + 4r^2 + 2k^2} \cdot x^2 + \frac{16Rr + 4r^2 + k^2}{2} \cdot x$$
$$- 2Rr\sqrt{16Rr + 4r^2 + 2k^2} = 0.$$

In the special case for which we put

$$R = \frac{71}{13}\sqrt{\frac{13}{43}}, \quad r = \sqrt{\frac{43}{52}}, \quad k^2 = 61,$$

the equation will be reduced to the form

$$x^3 - 13x^2 + 54x - 71 = 0,$$

or

$$(x-3)(x-4)(x-6) + 1 = 0.$$

The three roots of this equation, α, β, γ, which are arranged in order of magnitudes, lie between the intervals

$$2 < \alpha < 3, \quad 4 < \beta < 5, \quad 5 < \gamma < 6;$$

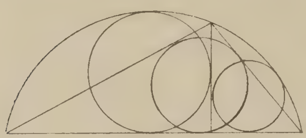
and none of them can have a rational value, for the roots of an equation, whose coefficient of the highest term is unity, can be rational numbers, when and only when they are integers positive or negative.

But the roots of a cubic cannot be constructed geometrically except when they are rational numbers.

Therefore the roots of the equation before us cannot be constructed by a geometrical means even in its special case. Our problem is therefore impossible in general to be geometrically solved.

Y. SAWAYAMA, ON A GEOMETRICAL THEOREM DEVICED BY THE OLD JAPANESE SCHOOL OF MATHEMATICS.

In the *Sampō Semmon-Shō*, compiled by Gokai in 1841 there stands a question that finds its enunciation in these words:



There is a circular segment, as shown in the figure, in which a triangle is inscribed and there are described three circles, a great, a middle, and a small. The diameter of the great circle is 4 inches, that of the small circle 2 inches. How long is the diameter of the middle circle?

The answer is given, in the work, as *the middle diameter = 3 inches*, while the author indicates his process of calculation thus: *The diameters of the great and small circles being added together, one half of this will give the diameter of the middle circle.*

There is added no proof or explanation in Gokai's work, as has been the case with many other publications of mathematicians in old Japan.

I.

Y. Sawayama takes up this problem and succeeds in establishing some theorems by extending the meaning of the proposition contained in the above enunciation of Gokai's. The result of his study is given in the *Journal of the Tokyo Physics School, Vol. 9, pp. 221—224* and *259—266, June* and *July,* 1900.

1. *If a perpendicular be let fall from the vertex to the base of a triangle whose base angles are both acute, and if the circles are described that touch the perpendicular and the base and the circumcircle of the triangle internally, and also those circles that touch the base produced and the perpendicular produced across the base and the circumcircle externally, then we can deduce certain propositions that concern to these circles.*

(1) *The arithmetic mean of the radii of the two inscribed circles is equal to the radius of the circle inscribed in the triangle, and the arithmetic mean of the radii of the two externally touching circles is equal to the radius of the escribed circle that touches the base.*

(2) *The geometric mean of the radii of the two circles that are situated in the two opposite angles formed by the base and the perpendicular*

is equal to half the least chord of the circumcircle that can be drawn through the foot of the perpendicular.

(3) *The centres of the four circles are concyclic.*

2. We first consider some lemmas that serve for the establishment of our propositions:

Lemma 1. A circle O is cut by a transversal BC and a circle is drawn through the points B and C, with the middle point M of the arc BC as centre. If then a third circle J is described touching the transversal on the opposite side (or on the same side) of the point M and also touching the circle O internally (or externally), the circles J and M intersect orthogonally; and vice versa.

For let P be the point where the circle J touches the transversal and Q the point where the circles J and O touch. Join OM cutting the transversal and the circle O in G and N respectively. From M draw MT touching the circle J.

From the two isosceles triangles OMQ, JPQ we see the collinearity of the three points Q, P, and M.

The angles PQN and PGN are each a right angle, so that the four points P, G, N, Q are concyclic.

Hence we infer that

$$MQ \cdot MP = MN \cdot MG, \quad \text{and} \quad MQ \cdot MP = MT^2;$$

also

$$MN \cdot MG = MB^2 = MC^2.$$

$$\therefore\ MT^2 = MB^2 = MC^2, \quad \text{or} \quad MT = MB = MC;$$

whence follows our lemma.

Lemma 2. If in the circle O in last lemma a transversal AA' is drawn cutting BC at right angles, A and A' being its points of intersection with the circumference, and if from the point I, in which MA or MA', produced if necessary, meets the circle, IE and IF are let fall perpendiculars on BC and AA' respectively, then the three points E, F and M are collinear.

The converse of this proposition is also true.

Lemma 3. The radius of the circle inscribed in a triangle ABC (or the escribed circle that touches the side BC) is greater than, equal to, or less than, the distance IF from the incentre (or excentre) I to the height AD, according as the non-smaller of the base angles is acute, right or obtuse.

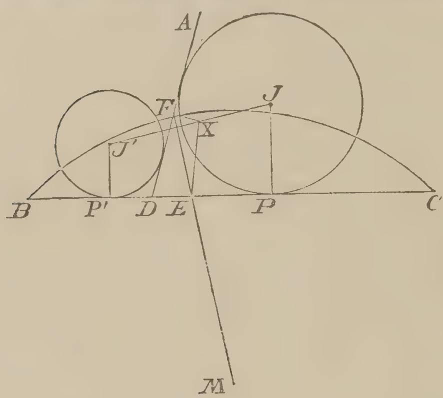
3. We now proceed to the demonstration of our propositions.

Proof to the proposition (1).

Let the base angles of a triangle be both acute, and let D be the foot of the perpendicular let fall from A on the base. Then the arithmetic mean of the radii of the two circles that touch BC and AD (or BC produced and AD produced through D) and the circumcircle internally (or externally) will be equal to the radius of the circle inscribed (or escribed touching the base BC) in the triangle.

We carry our proof in four stages.

(A) *If from the middle point X of the line joining the centres of the two circles J and J′ that touch AD and BC, which intersect in D, and that are described orthogonally intersecting with a circle M, which cuts BC in the points B and C, perpendiculars XE and XF are let fall on BC and AD, then the three points E, F and M will be collinear.*



For as X is the middle point of JJ', so is E the middle point of PP', where P and P' are the points of contact of the circles with BC. Hence we have

$$EJ^2 \sim EJ'^2 = (EP^2 + PJ^2) \sim (EP'^2 + P'J'^2) = PJ'^2 \sim P'J'^2.$$

From the orthogonality of the circles J and J' with M, we deduce

$$MJ^2 = PJ^2 + MB^2, \quad MJ'^2 = P'J'^2 + MB^2,$$

$$\therefore\ MJ^2 \sim MJ'^2 = PJ^2 \sim P'J'^2,$$

whence

$$EJ^2 \sim EJ'^2 = MJ^2 \sim MJ'^2,$$

whence again we infer that ME and JJ' are at right angles.

MF and JJ' are also at right angles in a similar manner.

Therefore the three points E, F and M are collinear.

(B) *If in* (A) *AD and BC are at right angles, the point X will be on the circle M.*

For in the triangle MJJ' we have

$$2 \cdot XJ^2 + 2 \cdot MX^2 = MJ^2 + MJ'^2.$$

Since the triangle constructed on JJ' as base by drawing its sides parallel to BC and AD has them equal to $\varrho + \varrho'$ and $\varrho - \varrho'$, where ϱ and ϱ' denote the radii of the circles J and J', so it is

$$2 \cdot XJ^2 = \tfrac{2}{4}\{(\varrho + \varrho')^2 + (\varrho - \varrho')^2\} = \varrho^2 + \varrho'^2.$$

This and the values

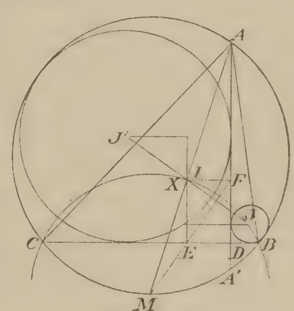
$$MJ^2 = \varrho^2 + MB^2, \quad MJ'^2 = \varrho'^2 + MB^2,$$

that arise from the orthogonality of the circles, being substituted in the first obtained expression, it becomes after reduction

$$2 \cdot MX^2 = 2 \cdot MB^2, \quad \text{or} \quad MX = MB.$$

The point X is therefore on the circle M.

(C) *Assuming that there are at least two circles that touch the base BC (or base produced) and the height AD (or its prolongation through D) and that touch the inscribed circle of the triangle internally (or externally the escribed circle touching the base), the straight line that joins their centres J and J' will pass through the incentre I of the triangle (or the excentre pertaining to the base) and will be at right angles to the line that goes through the feet E, F of the perpendiculars let fall from I on BC and AD; and the distances from the centres to IE will be each equal to the radius of the inscribed (or escribed) circle.*



First to prove that the middle point X of the line JJ' is coincident with the point I, and consequently that the points J and J' are equidistant from the line IE.

Describe the circle BIC, and denote its centre by M. By Lemma 1 the circles J and J' cut this circle at right angles.

Since the circles J and J' touch BC and AD and intersect with the circle BIC orthogonally, the feet E' and F' of the perpendiculars let fall from the point X on BC and AD are collinear with the centre M of the circle BIC, as we have proved in (A).

Since the circles J and J' touch BC and AD, which are at right angles, and cut orthogonally the circle BIC, we infer by (B) that the point X is on the circle BIC.

Since X is on the circle M, and the points E', F' and M are collinear, X is by Lemma 2 on the line AM or on the line that joins M with the point A' where AD intersects with the circumcircle again.

It is evident that one of the two intersections of AM with the circle BIC is the incentre of the triangle ABC and the other the excentre that appertains the side BC, and that the intersections of $A'M$ and the circle are the excentres of the triangle $A'BC$ corresponding to the sides $A'B$ and $A'C$.

Hence to show that X is the incentre of the triangle ABC (or the excentre in regard to the side BC), it is only necessary to point out that this point is not the intersection of $A'M$ with the circle BIC.

As the base angles of the triangle ABC are both acute, the point D is within the base. The two angles $A'BC$ and $A'CB$ of the triangle $A'BC$ are also acute. Hence the distance to BC from the intersection X' of $A'M$ and the circle BIC is less than that from X' to AA', as will be easily proved.

But the distance from the point X to BC is greater than that to AA'. For the former is equal to the sum of the radii of two inscribed circles, while the latter is equal to their difference.

The point X, therefore, cannot but coincide with I.

Next we shall prove that the points J and J' are each on the straight line that passes through I and is normal to EF, and that the distances from IE are equal to the radius of the inscribed circle (or the circle escribed to the side BC) of the triangle ABC.

From what we have said in the demonstration of (A), JJ' is at right angles with $E'F'$. But the point X being coincident with I, the points E and F must coincide with E' and F' respectively. Hence JJ' is perpendicular to EF. The radii of the circles J, J' and I, that are drawn to their point of contact with BC are all parallel, and I is the middle point of JJ', so is E the middle point of the part of BC that lies between the points of contact of J and J', that is, the middle point of a segment equal to the sum of the radii of J and J'. Therefore the distances from J and J' to IE are both equal to the finite IE.

(D) *There are two and only two inscribed (or externally touching) circles spoken of in* (C).

If we assume that there are inscribed (or escribed) circles on each side of the height AD of the triangle ABC, the number of these circles is evidently not more than one by what we have said in (C).

Next we have to prove that there is always such a circle on each side of AD.

Let J_1 and J_2 be the two points that lie on the line perpendicular to EF and passing through the point I, spoken of in (C), such that the distances from the line IE are equal to the finite IE. Let also J_1 and C be on the same side of IE and let the angle ABC be not less than the angle ACB.

From J_1 and J_2 draw J_1P_1 and J_2P_2 at right angles to BC, and draw J_1V_1 and J_2V_2 perpendicular to AD.

In the two right angled triangles EFI and J_2IW, where W is the intersection of IE and J_2V_2, the two sides EF and EI of the former are respectively normal to J_2I and J_2W, so the acute angles FEI and IJ_2W are equal; and $IE = J_2W$ by construction. Hence the triangles are similar, and we have

$$IE = WJ_2, \quad IF = WI.$$

Evidently $IF = WV_2$, too.

But by Lemma 3, IE is $> IF$. Hence IE and WJ_2 are respectively $> IW$ and WV_2.

The points J_1 and J_2 are therefore on the same side of BC and on opposite sides of AD.

Since EI, EP_1, EP_2 are equal with each other by construction, the circle described with E as centre and EI as radius touches J_1P_1 and J_2P_2 at P_1 and P_2.

Since P_1J_1, EI and P_2J_2 are parallel to one another and E is the middle point of P_1P_2, so I is the middle of J_1J_2. Hence F is the middle point of V_1V_2. The circle with F as centre and FI as radius touches therefore J_1V_1 and J_2V_2 at V_1 and V_2.

Since the line J_1J_2 passes through I, which is one of the intersections of the three circles P_1IP_2, V_1IV_2, BIC, and is perpendicular to the line on which the three centres E, F and M lie, this line is a secant common to these circles.

The tangents drawn from a point on J_1J_2 to these three circles are therefore equal, as is treated in ordinary works on geometry.

Hence the circle described with J_1 as centre and J_1P_1 as radius touches BC and AD and intersects with the circle BIC orthogonally. If, therefore, J_1 and M be on opposite sides (or on the same side) of BC, by Lemma 1, the circle J_1 touches the circle ABC internally (or externally).

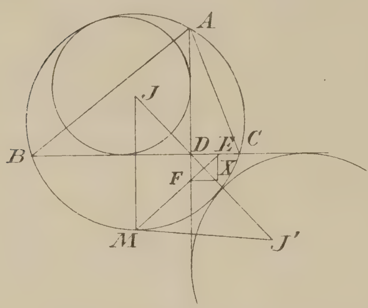
In a similar way the circle with J_2 as centre and with J_2P_2 as radius orthogonally intersects with the circle BIC and consequently touches the circle ABC internally (or externally).

4. *Proof to the second proposition.*

Let D be the foot of the perpendicular let fall from the vertex on the base of a triangle ABC with acute angles at B and C. Let J and J' be the centres of the circles described to touch BC and AD and the circumcircle of the triangle internally and externally respectively, on the opposite sides of AD. Let DL be half the least chord of the circle ABC that passes through D.

In this case the geometric mean of the radii of the circles J and J' will be equal to DL.

Let X be the middle point of JJ' and M the centre of the circle BIC. Then we have from the triangle JMJ'



$$2\cdot XJ^2 + 2\cdot MX^2 = MJ^2 + MJ'^2.$$

Again, since JJ' is the hypothenuse of a right-angled isosceles triangle whose sides are $= \varrho + \varrho'$, where ϱ and ϱ' denote the radii of the circles J and J', we have

$$2\cdot XJ^2 = (\varrho+\varrho')^2 = \varrho^2 + \varrho'^2 + 2\varrho\varrho'.$$

Since J and J' orthogonally intersect with the circle BIC, by Lemma 1, we have

$$MJ^2 = \varrho^2 + MB^2, \quad MJ'^2 = \varrho'^2 + MB^2.$$

These values being substituted, the first obtained expression becomes, after reduction,

$$\varrho\varrho' = MB^2 - MX^2.$$

Now $MX = MD$, for the perpendiculars XE and XF let fall from X on BC and AD are equal, because both are equal to half the difference of ϱ and ϱ'. Hence $FD = FX$, and by what we have said in the proof to the first proposition, MF passes through E and is perpendicular to JJ', that is, perpendicular to DX. In other words, FE is the perpendicular bisector of the base XD of the isosceles triangle FDX.

We have therefore

$$MB^2 - MD^2 = DB\cdot DC, \quad \text{and} \quad DB\cdot DC = DL^2,$$
$$\therefore\ \varrho\varrho' = DL^2.$$

5. *To prove the proposition* (3).

Let J_1 and J_2 be the centres of the circles that touch the base BC and the height AD of a triangle ABC with acute angles B and C, and touch internally the circumcircle; further let J_3 and J_4 be the centres of the circles that touch BC and AD, produced, and the circumcircle externally. Let also J_1 and J_4 be on opposite sides of AD. Then the four points J_1, J_2, J_3, J_4 will be concyclic.

Denoting the radii of the four circles by ϱ_1, ϱ_2, ϱ_3, ϱ_4, we have, by what we have said,

$$\varrho_1\varrho_2 = \varrho_2\varrho_3 = DL^2, \quad \text{or} \quad \varrho_1 : \varrho_2 = \varrho_3 : \varrho_4.$$

As DJ_1, DJ_2, etc., are the diagonals of the squares with ϱ_1, ϱ_2, etc., as sides, we have

$$\varrho_1 : \varrho_2 = DJ_1 : DJ_2, \quad \varrho_3 : \varrho_4 = DJ_3 : DJ_4,$$
$$\therefore\ DJ_1 : DJ_2 = DJ_3 : DJ_4.$$

Also the angles J_1DJ_3 and J_2DJ_4 are equal, so that the triangles J_1DJ_3 and J_2DJ_4 are similar, and we infer the equalness of the angles $J_1J_3J_2$ and $J_1J_4J_2$.

Therefore the four points J_1, J_2, J_3, J_4 are concyclic.

II.

M. Endō's study.[1])

The same problem in the *Sampō Semmon-Shō*, that was considered by Y. Sawayama, was also studied by M. Endō, who published his result soon after Sawayama's preceding paper had appeared.[1]) The following is the result arrived at by M. Endō.

If we describe two circles that touch the altitude AD and the base BC of a triangle ABC, whose base angles B and C are both acute, and that touch internally the circumcircle of the triangle, then the middle point J of the straight line, that joins their centres i_1 and i_2, is the incentre of the triangle; and the middle point J' of the centre-line $i_3' i_4'$ of the circles described touching the prolongations of the altitude and the base and also touching externally the circumcircle is the excentre that belongs to the side BC.

For, let O, I and I' be the circumcentre, the incentre and the excentre within the angle A; let G and F be the points where AI' meets BC and the circumcircle; from F draw the diameter FH, cutting again the circumference in H; from the points A, I, G, I' let fall on the tangent at F the perpendiculars AE, IM, GL, $I'M'$; draw OX at right angles to AE; join FB; and denote XO, XD, XA and the circumradius by a, b, c and R. We then have

$$IF^2 \text{ (and } I'F^2) = FB^2 = FK \cdot FH = FG \cdot FA,$$

$$(1) \qquad \therefore\ IM^2 \text{ (and } I'M'^2) = GL \cdot AE = (R-b)(R+c).$$

Again

$$\frac{FL}{FE} = \frac{GL}{AE}, \quad \text{or} \quad \frac{FL}{a} = \frac{R-b}{R+c}.$$

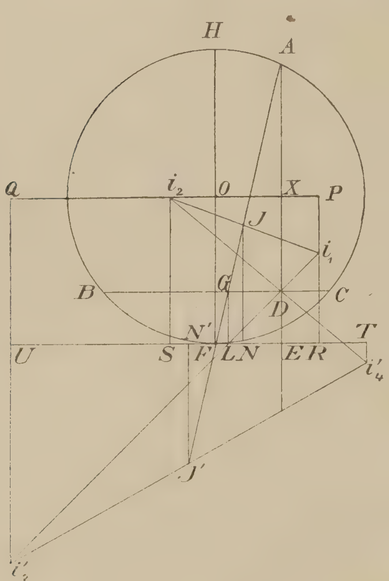
1) Journal of the Phys. School in Tokyo, Vol. 9, pp. 294—297, August, 1900.

Consequently

$$(2) \quad MF^2 \text{ (and } M'F^2) = FL\cdot FE = \frac{a^2(R-b)}{R+c}$$
$$= \frac{(R^2-c^2)(R-b)}{R+c} = (R-b)(R-c).$$

(3) If the angle B is greater than C, so I is to the right of FH and upward of FE; I' to the left of FH and downward of FE.

Let r_1 and r_2, ϱ_3 and ϱ_4 be the radii of the circles i_1 and i_2, i_3' and i_4'.



If then we make a construction as shown in the figure, we shall have

$$Oi_1^2 = OP^2 + Pi_1^2,$$

or

$$(R-r_1)^2 = (a+r_1)^2 + (b-r_1)^2,$$

whence

$$r_1 + (a-b+R) = \sqrt{2(R+a)(R-b)}.$$

Similarly

$$r_2 + (R-a-b) = \sqrt{2(R-a)(R-b)}.$$

Consequently

$$(4) \quad JN = \tfrac{1}{2}(i_1R + i_2S)$$
$$= \tfrac{1}{2}\{(R-b+r_1)+(R-b+r_2)\}$$
$$= \sqrt{(R-b)(R-c)},$$

and

$$(5) \quad FN = \tfrac{1}{2}(FR-SF) = \tfrac{1}{2}\{(a+r_1)-(r_2-a)\} = \sqrt{(R-b)(R-c)}.$$

(6) J is upward of FE, and since $FR > FS$, N is to the right of F.

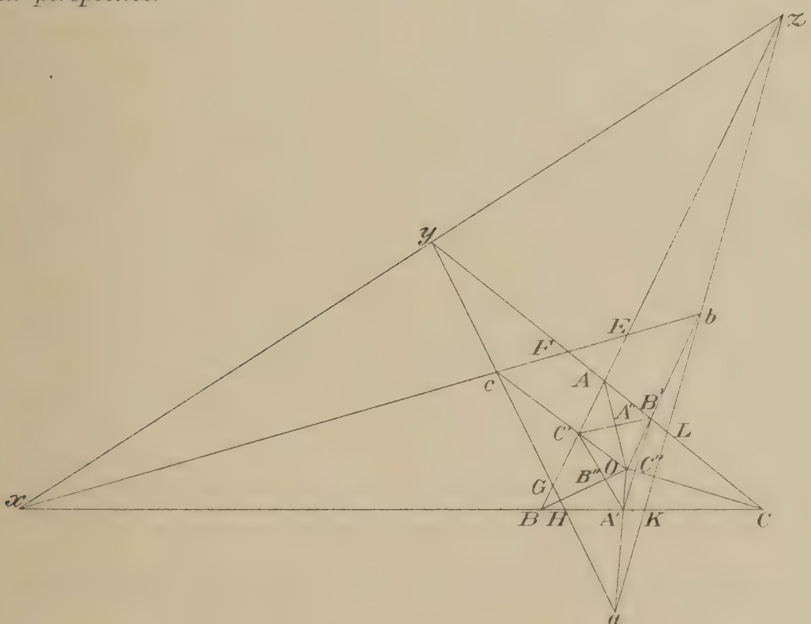
By comparing (4), (5), (6) with (1), (2) and (3), we see that J coincides with I.

In like manner J' and I' are also coincident with one another.

Y. SAWAYAMA, ON THREE TRIANGLES THAT ARE IN PERSPECTIVE TWO BY TWO.[1]

1. *A triangle $A'B'C'$ has its vertices on the three sides of another triangle ABC, and a third triangle $A''B''C''$ has its vertices on the sides of $A'B'C'$. When ABC and $A'B'C'$, and so also $A'B'C'$ and $A''B''C''$ are in perspective, ABC and $A''B''C''$ are also in perspective.*

If now the triangles $A'B'C'$ and $A''B''C''$ have their centre of perspective in the centre of gravity of the triangle $A'B'C'$, and if there be a fourth triangle abc whose three sides are respectively parallel to those of $A'B'C'$, and if the centre of perspective of these latter two be identical with that of ABC and $A''B''C''$, then ABC and abc will be in perspective.



For, let P, Q, R be the points where AA'', BB'', CC'' intersect with BC, CA, AB, respectively. The intersection of AP with BB' will be denoted by D.

The line AP being considered as a transversal in the triangles $B'BC$ and $C'BB'$, we have

1) The Journal of Phys. Sch. in Tokyo, Vol. 13, pp. 3—5, June, 1904.

$$\frac{BP}{CP}=\frac{BD}{B'D}\cdot\frac{B'A}{CA},\quad \frac{BD}{B'D}=\frac{BA}{C'A}\cdot\frac{C'A''}{B'A''},$$

$$\therefore\ \frac{BP}{CP}=\frac{BA}{C'A}\cdot\frac{C'A''}{B'A''}\cdot\frac{B'A}{CA}=-\frac{C'A''}{B'A''}\cdot\frac{AB'}{AC'}\cdot\frac{AB}{CA}.$$

Similarly we have

$$\frac{CQ}{AQ}=-\frac{A'B''}{C'B''}\cdot\frac{BC'}{BA'}\cdot\frac{BC}{AB},\quad \frac{AR}{BR}=-\frac{B'C''}{A'C''}\cdot\frac{CA'}{CB'}\cdot\frac{CA}{BC}.$$

From these relations it follows

$$\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}$$
$$=-\left(\frac{C'A''}{B'A''}\cdot\frac{A'B''}{C'B''}\cdot\frac{B'C''}{A'C''}\right)\times\left(\frac{AB'}{CB'}\cdot\frac{BC'}{AC'}\cdot\frac{CA'}{BA'}\right)\times\left(\frac{AB}{CA}\cdot\frac{BC}{AB}\cdot\frac{CA}{BC}\right)$$
$$=-(-1)(-1)\cdot 1=-1.$$

Therefore AP, BQ, CR pass through a point, and thus our maintenance is established.

2. Next we shall see the relation that exists when A'', B'', C'' are specially *the middle points of the sides of the triangle* $A'B'C'$.

Let O be the centre of perspective of the triangles ABC and $A''B''C''$; let x, y, z be the intersections of bc, ca and ab with BC, CA and AB respectively; let bc intersect CA and AB in E and F, and ca with AB and BC in G and H, and ab with BC and CA in K and L.

Since bc and AF are each bisected by AO, so we have $bE=Fc$, and similarly $cG=Ha$; and $aK=Lb$.

Again from the transversal BC, CA, AB of the triangle abc, we have

$$\frac{bx}{cx}=\frac{bK}{aK}\cdot\frac{aH}{cH},$$

$$\frac{cy}{ay}=\frac{cE}{bE}\cdot\frac{bL}{aL}=\frac{cE}{bE}\cdot\frac{aK}{bK},$$

$$\frac{az}{bz}=\frac{aG}{cG}\cdot\frac{cF}{bF}=\frac{cH}{aH}\cdot\frac{bE}{cE},$$

whence we get by multiplication

$$\frac{bx}{cx}\cdot\frac{cy}{ay}\cdot\frac{az}{bz}=1.$$

It follows therefore that *the three points x, y, z are collinear.*

Y. SAWAYAMA, ON A GEOMETRICAL THEOREM.[1])

1. As is generally known, *if we describe two circles, whose radii are R and r respectively, circumscribing about and inscribing in a triangle, there exists the relation*

$$\mu + \mu' + \mu'' = 2R - r,$$

where μ, μ' and μ'' denote the sagittae of the three segments of the circumcircle made by the sides of the triangle.

This theorem can be generalised to the case of a polygon.

For, let $A_0 A_1 A_2 \ldots A_{n-1}$ be an n-gon inscribed in a circle, and let the sagitta of the outward segment made by the side $A_{p-1}A_p$ be μ_p, those of the segments on the diagonal $A_0 A_{p+1}$ and containing A_1 and A_{n-1} be respectively, δ_p and d_p, and the radius of the circle inscribed in the triangle $A_0 A_p A_{p+1}$ be r_p. We have then from our original theorem

$$\begin{aligned}
&\mu_1 + \mu_2 + d_1 = 2R - r_1, \\
&\delta_1 + \mu_3 + d_2 = 2R - r_2, \\
&\ldots\ldots\ldots\ldots\ldots \\
&\delta_{n-3} + \mu_{n-1} + \mu_n = 2R - r_{n-2}.
\end{aligned}$$

By adding together these identities and taking into notice the relation $d_p + \delta_p = 2R$, we obtain

$$(1) \qquad \mu_1 + \mu_2 + \ldots + \mu_n = 2R - (r_1 + r_2 + \ldots + r_{n-2}).$$

Hence follows the generalised theorem: *If we draw the diagonals from one of the vertices of a polygon inscribed in a circle of radius R and denote the sum of the radii of the circles inscribing the triangles thus formed by Σr, so also the sum of the sagittae of the outward segments made by the sides of the polygon by $\Sigma\mu$, then there exists the relation*

$$\Sigma\mu = 2R - \Sigma r.$$

From this theorem naturally follows the Chinese theorem made public by Y. Mikami.[2])

1) Journal of Phys. Sch. in Tokyo, Vol. 15, pp. 362—365, September, 1906.

2) A Chinese theorem on geometry. Archiv der Mathematik und Physik, III. Reihe, Bd. IX, S. 309—310. The theorem runs thus: If in a polygon inscribed in a circle all possible diagonals that can be drawn from a vertex are drawn and the successive triangles thus formed are inscribed with circles, then their radii will be together equal for any of the vertices.

This theorem can be easily established, by mathematical induction, when it is proved for the case of an inscribed quadrilateral $ABCD$.

2. Something more may be said with the same proposition.

i. The sum of the ex-radii of the triangles considered above is also constant for any vertex.

If r', r'', r''' denote the ex-radii of the triangle inscribed in a circle, then

$$r' + r'' + r''' = 4R + r,$$

as we know, and like formulae being formed for the $n-2$ triangles, and these being added together, we deduce

$$\Sigma r' = 4(n-2)R + \Sigma r.$$

In that case, the sides AB, BC, CD and DA will have such expressions like $AB = 2R\sin\alpha$, α, β, γ, δ denoting the angles subtended by these sides, and R the circumradius. Thus the formulae for the radii of the circles inscribed in the triangles ABC and ABD, expressed in terms of the sides of the quadrilateral, reduce to

$$\varrho = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma+\delta}{2}, \quad \varrho' = 4R\sin\frac{\gamma}{2}\sin\frac{\delta}{2}\sin\frac{\alpha+\beta}{2},$$

so that their sum becomes a symmetrical expression of its arguments. Therefore the validity of our proposition follows.

In May, 1906, three different proofs of the same theorem were published by T. Hayashi in the Journal of the Physics School in Tokyo, and afterwards also in the Mathesis (3) VI, where two more are added, Sawayama's proof inclusive.

The following proofs are taken from Hayashi's writings.

Nagasawa's proof. Let O be the circumcentre, r_1, r_2, r_3, r_4 the radii of the circles I_1, I_2, I_3, I_4 inscribed in the triangles ABC, BCD, CDA, DAB, and a, b, c, d, e, f the perpendiculars drawn from O to AB, BC, CD, DA, AC, BD. Then by a known theorem

$$R + r_1 = a + b + e, \quad R + r_3 = c + d - e,$$
$$\therefore\ 2R + r_1 + r_3 = a + b + c + d.$$

A similar way leads to the same expression for $2R + r_2 + r_4$, and therefore $r_1 + r_3 = r_2 + r_4$.

Nozaki's method. The quadrilateral $I_1 I_2 I_3 I_4$ is a rectangle. Taking S for its centre, we have

$$OI_1^2 + OI_3^2 = 4\cdot OS^2 + 4\cdot I_1S^2 = 4\cdot OS^2 + 4\cdot I_4S^2 = OI_2^2 + OI_3^2.$$

But $OI_1^2 = R^2 - 2Rr_1$, etc, so that $r_1 + r_3 = r_2 + r_4$.

Matsuo and Ōmori's demonstration. Let V and W be the intersections of AC and I_1I_3, BD and I_2I_4. Then since

$$\angle I_3AV = I_2BW, \quad I_4I_3V = I_1I_2W,$$
$$\angle AI_3I_4 = ADE = BCE = BI_2I_1,$$

the triangles AI_3V and BI_2W are similar. Hence the angles included by the two pairs of the lines are equal. Consequently, since $I_1I_3 = I_2I_4$, the orthogonal projections of these lines respectively on the perpendiculars to AC and BD are equal; but these projections are equal the one to $r_1 + r_3$ and the other to $r_2 + r_4$.

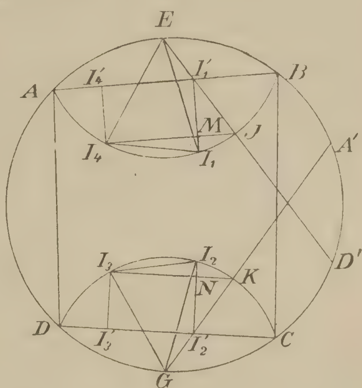
ii. The sum of the in-radii of the triangles under description for a regular polygon inscribed in a circle is greater than the same quantity for any other polygon with same number of sides. The same may also be said with ex-radii.

Chou's proof. Project I_1, I_4 in I_1', I_4' on AB, and I_3, I_2 in I_3', I_2' on CD: then $I_1 I_1' = r_1$, etc.

E and G being the middle of the arcs AB and CD, take the arcs $EBD' = EAD$, $GCA' = GDA$. The chord ED' cuts the arc $BI_1 I_4 A$ in a point J, and the chord GA' cuts the arc $CI_2 I_3 D$ in K.

Let JI_4 and $I_1 I_1'$, KI_3 and $I_2 I_2'$ intersect in M and N. Then

$$\angle I_1 I_4 J = \tfrac{1}{2} I_1 EJ = \tfrac{1}{2} CED',$$
$$\angle I_2 I_3 K = \tfrac{1}{2} I_2 GK = \tfrac{1}{2} BGA'.$$



But since the arcs BA' and CD' are equal, the angles $I_2 I_3 K$ and $I_1 I_2 J$ are equal, and hence the right-angled triangles $I_1 I_4 M$ and $I_2 I_3 N$ are similar.

Denoting AB, BC, CD, DA, AC, BD by a, b, c, d, e, f, we have

$$AI_1' = \tfrac{1}{2}(a+e-b), \quad AI_4' = \tfrac{1}{2}(c+d-f),$$
$$\therefore \; I_4' I_1' = I_4 M = \tfrac{1}{2}(e+f-b-d),$$

and similarly we obtain the same expression for $I_2' I_3' = I_3 N$. Consequently the triangles $I_1 I_4 M$, $I_2 I_3 N$ are equal, so that $I_1 M = I_2 N$, that is,

$$r_1 - r_4 = r_2 - r_3, \quad \text{or} \quad r_1 + r_3 = r_2 + r_4.$$

The above theorem is said to have come from Chou Ta, who is one of the leading mathematicians in present China. But as T. Hayashi observes, the theorem for the case of a quadrilateral is given in the Zoku Shimpeki Sampo, or Mathematical Problems Suspended Before Temples, Second Series, collected by Fujita Kagen, 1806, Appendix, Sheet 5, and it was K. Nagasawa who achieved a proof of it and communicated to Chou.

The theorem, as contained in the old Japanese publication mentioned above, it stands thus:

There is a circle, as in the figure (here we omit it), in which six lines are drawn and four circles are inscribed (each touching three of the lines). The S diameter 1 inch, the E diameter 2 inches, the W diameter 3 inches. Required the length of the N diameter.

Answer. The N diameter 4 inches.

Direction. Arrange the E diameter, add the W diameter, subtract the S diameter, then the remaining is the N diameter, and thus the problem is solved.

Yedo, Fifth Month, 1800.

Maruyama Tetsugorō Ryōkwan,
Pupil to Maruyama Ryōgen.

For take M_p for the middle point of the arc $A_{p-1}A_p$. Then $2R\cdot\mu_p = M_pA_p^2$, so that

$$(2)\qquad 2R\sum_{p=1}^{n}\mu_p = \sum_{1}^{n} M_pA_p^2.$$

The right-hand side represents half the sum of the squares of the sides of the $2n$-gon whose vertices are the points $A_0, A_1, A_2 \ldots A_{n-1}$ and the middle points of the arcs opposite to these points.

Now if the two consecutive sides $A_{p-1}A_p$ and A_pA_{p+1} of the original n-gon are unequal, we exchange the vertex A_p with a_p the middle point of the arc $A_{p-1}A_{p+1}$ and form a polygon with the same number of sides. Let m_p and m_{p+1} be the middle points of the arcs $A_{p-1}a_p$ and a_pA_{p+1}. As the arcs of the segments $M_pA_pM_{p+1}$ and $m_pa_pm_{p+1}$ are each equal to half the arc $A_{p-1}A_{p+1}$, these two segments are congruent; and as the angles contained in these are obtuse, the median passing through the vertex A_p of the triangle $M_pA_pM_{p+1}$ is greater than that through a_p of the triangle $m_pa_pm_{p+1}$; so that

$$M_pA_p^2 + A_pM_{p+1}^2 > m_pa_p^2 + a_pm_{p+1}^2.$$

Hence for the $2n$-gon obtained by replacing $A_{p-1}m_p$, m_pa_p, a_pm_{p+1}, $m_{p+1}A_{p+1}$ for $A_{p-1}M_p$, M_pA_p, A_pM_{p+1}, $M_{p+1}A_{p+1}$ in the original $2n$-gon, the sum of the squares of the sides is less than the corresponding sum for the original polygon. Therefore the sum of the sagittae for the new n-gon is less than that for the original n-gon by (2), so that by (1) follows our proposition.

iii. Of the sums of in-radii under consideration for two regular polygons inscribed in the same circle, that is greater which pertains to the polygon with greater number of sides. The same consists also of ex-radii.

For if we form an $(n+1)$-gon by replacing the side $A_{p-1}A_p$ of a regular n-gon inscribed in a circle by the chords of the arcs $A_{p-1}M_p$, M_pA_p, which are halves of the arc $A_{p-1}A_p$, and denote the middle points of these half arcs by B and C respectively, we have

$$A_{p-1}B^2 + BM_p^2 < A_{p-1}M_p^2,$$

since the angle $A_{p-1}BM_p$ is obtuse.

Hence the sum of the squares on the sides of the $2n$-gon with vertices in the vertices of the original regular n-gon and the middle points of the arcs subtended by its sides is greater than the same sum for the $2(n+1)$-gon constructed for the new formed $(n+1)$-gon. Hence by (1) and (2) our proposition becomes established.

Y. SAWAYAMA, ONE OF MANNHEIM'S THEOREMS EXTENDED.[1]

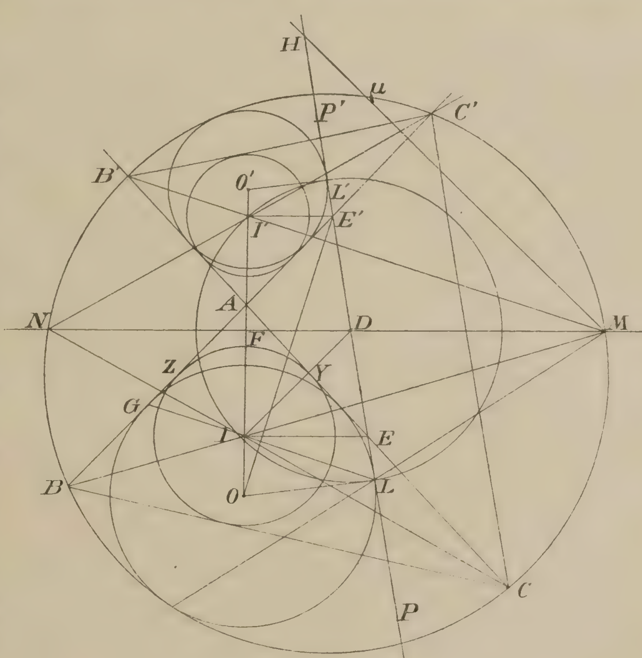
The envelope of the circle circumscribed about a triangle, whose two sides and inscribed circle are fixed in position, is a circle.

This theorem is one that dues to Mannheim and a well-known one. Its meaning may be extended in one way, and the proposition thus obtained runs as follows:

When a straight line turns round a given circle that touches two given straight lines, the envelope of the circle described on the intercept of the turning line, cut off by the two given lines, as chord, so as the segment on one side of it contains a constant angle, is two circles.

Let AY and AZ be the two given lines and I the centre of the given circle, and we denote by BC the moving line that touches the circle I. A circle is constructed on BC as chord, so that the segment on one side of this chord should always contain a constant angle (the said segment should lie on the same or opposite side of the chord as the circle I, according as I inscribes or escribes the triangle ABC).

The triangles ABC and $AB'C'$, where B' and C' are the points in which this circle meets AC and AB again, are similar.



Take I' for the point homologous to I with respect to the triangles $AB'C'$ and ABC, then $AI' : AI = AC' : AC$.

1) Journal of Phys. School in Tokyo, Vol. 15, pp. 569—572, November 1906.

But two angles of the triangle ACC' being constant, these ratios are of a constant value, and in consequence I' is a fixed point.

BI and $B'I'$ both pass through the middle point M of the arc CC', and CI and $C'I'$ pass the middle point N of the arc BB'.

As the triangles ABI and $AB'I'$ are similar, the angles MII' and $MI'I$ are equal, so that $MI = MI'$, and similarly $NI = NI'$.

Hence MN is the perpendicular bisector of II' and is in a fixed position.

From the fixed point I draw DI parallel to AB, and let it meet with MN in D, which is a fixed point. Draw PP' through D parallel to CC'. Since the triangle ACC' has two angles constant, the straight line CC' does not change its direction. Hence PP' is in a fixed position.

Since PP' is parallel to the chord of the arc CC' whose middle point is M, the moving circle has its centre on the perpendicular let fall from M to PP'.

The two angles MID and MNI are both equal to half an angle ZBC (where Z is the point of contact of the circle I on AB), so that the triangles MID and MNI are similar. Hence $MI^2 = MD \cdot MN$.

Now take M for the centre and MI^2 for the modulus of inversion, then the curve inverse to the fixed straight line PP' is the moving circle.

Hence the moving circle touches always the two circles that touch PP' among the radical circles with I and I' for limiting points.

Cor. 1. The two circles, that form the envelope of the moving circle, touch the two fixed straight lines AY and AZ.

For, draw IE and $I'E'$ perpendicular to II' and let them meet AC and AB in E and E' respectively. From the similarity of the triangles IFD, AIE, $AI'E'$, we have

$$FD : IF = IE : AI = I'E' : AI'.$$

Hence the half of II' which is the sum or difference of AI and AI' being equal to IF, we see from above proportion that half sum or half difference of IE and $I'E'$ is equal to FD.

But the distance between F and the intersection of FD and EE' is also equal to the above length, so D lies on EE'.

E and E' being homologous points of similar triangles ABC and $AB'C'$, we have $AE : AE' = AC : AC'$. Hence EE' is parallel to CC'. Consequently EE' is identical with PP', because both pass through D and are parallel to CC'.

Let O, O' be the centres of the two envelope circles, and let O lie on the same side of I' as I, and let the circles O, O' touch the line PP' in L and L'. Then the circle with D as centre and DI as radius goes through I', L, L' and touches the lines OL and $O'L'$.

Hence the line $I'E'$ is the polar of I with regard to the circle O and the line LE' the polar of L with respect to the same circle. So the point E', intersection of these lines, is the pole of the line that joins I and L. Hence OE' and IL are mutually at right angles.

Now denoting with G the intersection of IL and AZ, the lines DL, DI are equal, and ID, GE' are parallel. Thus $E'LG$ is an isosceles triangle.

Hence the line OE' bisects the angle $LE'G$.

Therefore the circle O touches the line AG, and consequently to the line AY too.

Similarly the circle O' touches the lines AY and AZ.

Cor. 2. The point of contact of one of the two envelope circles, that has the centre on opposite side of I as A, the point I and the middle point of the arc BC of the moving circle are collinear. Also the point of contact of the other circle having the centre on the same side of I as A, the point I and the middle point of the conjugate arc of BC are collinear.

We try the proof for the first proposition enunciated here.

Let μ be the middle point of the arc BC, and let the straight lines $M\mu$ and PP' intersect in H.

The angle IDL is equal to the angle contained in the arc BC, which we term K. Hence the angle ILL' is equal to half the supplement of K, and the angle $IM\mu$ is either equal to this or to its supplemental angle. Therefore the four points M, L, I, H are concyclic.

Hence taking M as the centre of inversion, the point inverse to L is the point of contact of the circle O with the moving circle, and those of I and H are I itself and μ, respectively.

For the establishment of the second proposition we have only to proceed in a similar way as in the first.

T. KARIYA, PROPERTIES OF THE TRIANGLE.[1]

I.

If we draw from the vertices of a triangle lines that contain equal angles, α say, with the opposite sides, the triangle formed by these lines will be in the ratio $4\cos^2\alpha : 1$ to the original triangle.

For let ABC be a given triangle. Draw AD, BE, CF, such that $\angle ADC = \angle BEA = \angle CFB = \alpha$.

Let GHI be the triangle formed by these lines.

The quadrilateral $BDIF$ can be inscribed in a circle, for one of its external angles is equal to its opposite internal angle. Hence the angles GIH and ABC are equal.

Similarly the angles IGH and GHI are equal to BCA and CAB respectively.

Hence the triangle HIG is similar to ABC.

Again we have

$$\frac{AB}{BG} = \frac{\sin(A+B)}{\sin(\alpha - B)} = \frac{\sin C}{\sin(\alpha - B)}, \qquad \frac{BC}{BH} = \frac{\sin A}{\sin(C - \alpha + A)},$$

so that

$$BH - BG = GH = \frac{\sin(C - \alpha + A)}{\sin A} \cdot BC - \frac{\sin(\alpha - B)}{\sin C} \cdot AB,$$

or after reduction,

$$= 2 \cdot AC \cos\alpha.$$

Therefore

$$\frac{\triangle ABC}{\triangle HIG} = \frac{AC^2}{GH^2} = \frac{1}{4\cos^2\alpha},$$

or

$$\triangle ABC : \triangle HIG = 1 : 4\cos^2\alpha.$$

II.

Take O for the centre of the circle inscribed in a triangle ABC, its points of contact with the sides being X, Y and Z. On OX, OY, OZ take D, E, F equidistant from O. Then the lines AD, BE, CF will meet in a point.

For, let these lines meet the opposite sides in J, H and K; and we have only to prove

$$BJ \cdot CH \cdot AK = JC \cdot HA \cdot KB,$$

or

$$(BX - x)(CY - y)(AZ + z) = (BX - z)(AZ + y)(CY + x),$$

where JX, HY, KZ are denoted by x, y, z respectively.

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1) Two papers in the Journal of the Phys. Sch. in Tokyo, Vol. 11, pp. 357—358, and Vol. 12, pp. 397—399, 1902 and 1903.

From A, B, C let fall on the opposite sides the perpendiculars AR, BS and CT; and denote the radius of the inscribed circle by r, and $OD = OE = OF$ by k. Then, since $BX = s - b$, $CY = s - c$, $AZ = s - a$, we have

$$\frac{XR}{x} = \frac{AR - (r-k)}{r-k},$$

$$\therefore \quad x = \frac{r-k}{AR - (r-k)} \cdot XR = \frac{a(r-k)}{2\triangle - (r-k)a}(c \cos B - s + b),$$

whence

$$BX - x = \frac{2\triangle\,(s-b) - ac\,(r-k)\cos B}{2\triangle - (r-k)\,a},$$

and similarly

$$CY - y = \frac{2\triangle\,(s-c) - ba\,(r-k)\cos C}{2\triangle - (r-k)b},$$

$$AZ + z = \frac{2\triangle\,(s-a) - bc\,(r-k)\cos A}{2\triangle - (r-k)\,c}.$$

We also form the expressions for $BX - z$, $AZ + y$, $CY + x$, the continued product of which will be evidently the same as the product of the three quantities above given. And thus our proposition follows.

For special values of r the following statements will be derived:

1^0. $k = r$. *The lines that join the vertices with opposite points of contact of the inscribed circle are concurrent.*

2^0. $k = \infty$. *The three perpendiculars are concurrent.*

3^0. $k = -r$. *If we denote the ends of the diameters that pass through X. Y and Z by X', Y' and Z', the three lines AX', BY' and CZ' are concurrent.*

ON A TRIANGLE WHOSE TWO BISECTORS OF ANGLES ARE EQUAL. (MIYATA AND HAYASHI.)

I.

The triangle, whose two bisectors of angles are equal. is an isosceles triangle.

This elementary theorem may be proved in various ways, of which the *Journal of the Tokyo Physics School* of February, 1901, contains six different kinds of demonstrations collected from various sources, and the April number in the same year further gives other three.

This problem was later taken up in the same *Journal* by Y. Miyata in his paper *on the properties of the bisectors of angles of a triangle.*[1])

One of his proofs is this:

In a triangle ABC, draw BD and CE bisecting its angles at B and C, and let these bisectors meet each other in the point O. Take up the triangle AEC and transpose it so as the points E and C fall on B and D respectively, and let the point A fall on the same side of BD as it originally lay; let it take the position of A', the line AO taking the position of $A'O'$.

The four points A, A', B, D lie on a circle, so that AO and $A'O$ intersect in the middle point M of the arc BD. Thus MA and MA' are equal chords, and they make equal angles with BD, namely, $\angle AOD = \angle A'O'B$, that is, $\angle AOD = \angle AOD = \angle AOE$. Hence the triangles AOD and AOE are congruent, and so $AD = AE$. Also it is evident that CD and BE are equal. It follows therefore the equalness of the sides AC and AB.

II.

Hereupon T. Hayashi studies the problem whether a triangle shall be isosceles or else not, if the bisectors of two external angles are equal.[2])

This problem would appear on the outset very easy of answer. But it is not so in actuality.

We denote the bisectors of the external angles at B and C in the triangle ABC by BD and CE. When the points C, D and B, E are both on the same sides of A, or when they are both on the opposite sides of A, we have nothing to say, for in these cases it is easy to prove the triangle to be isosceles.

But consider the case where C and D lie to the same side, and B and E on opposite sides of A. In this case the triangle cannot be concluded at once to be isosceles.

In the triangles ABD and BCE, we have

$$\frac{BD}{c} = \frac{\sin A}{\sin\left(A + B + \frac{180^\circ - B}{2}\right)} = \frac{\sin A}{\cos\left(A + \frac{B}{2}\right)},$$

$$\frac{CF}{a} = \frac{\sin B}{\cos\left(B + \frac{C}{2}\right)}.$$

1) Vol. 13, pp. 268—274, July, 1904.

2) Journ. of the Phys. School in Tokyo, Vol. 13, pp. 383—386, October, 1904.

Hence under the hypothesis that BD and CE are equal, we have

$$\frac{\sin C}{\cos\left(A+\frac{B}{2}\right)}=\frac{\sin B}{\cos\left(B+\frac{C}{2}\right)}.$$

But

$$\sin C\cos\left(B+\frac{C}{2}\right)-\sin B\cos\left(A+\frac{B}{2}\right)=\sin C\cos\left(B+\frac{C}{2}\right)+\sin B\cos\left(C+\frac{B}{2}\right)$$
$$=2\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\left(\sin^2\frac{A}{2}-\sin\frac{B}{2}\sin\frac{C}{2}\right),$$

and evidently $\sin\frac{B}{2}+\sin\frac{C}{2}>0$, so that

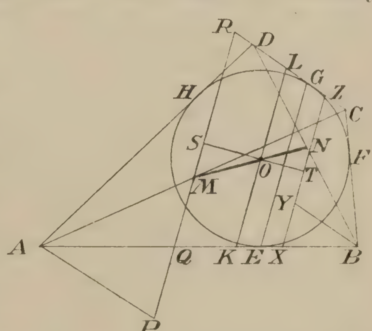
$$\sin^2\frac{A}{2}=\sin\frac{B}{2}\sin\frac{C}{2}.$$

Above transformations are all traceable backwards. We get therefore the

Proposition. If the sine of half an angle of a triangle is the mean proportional between the sines of half other angles, then the bisectors of the external angles of these angles are equal, but this triangle is not necessarily isosceles. When it is isosceles it will be an equilateral triangle, and the bisectors will be infinitely great.

Y. MIYATA, A PROOF OF A GEOMETRICAL THEOREM.[1])

The theorem: *The middle points of the diagonals of a quadrilateral circumscribed about a circle are collinear with the centre,* is usually demonstrated by the proposition: *The locus of the vertex common to two triangles that stand on given bases and that have a constant sum of areas is a straight line.*



But the latter proposition is somewhat difficult to prove. We therefore propose to demonstrate the former theorem in a more elementary way.

Let E, F, G, H be the points of contact. Through M, N, the middle of AC and BD, draw PQR, $XYZ \parallel EG$, cutting AB, CD in Q, R

1) Journ. of the Phys. School in Tokyo, Vol. 16, pp. 50—52, 1907.

and X, Z respectively and meeting respectively in P and Y the lines drawn from A and B parallel to CD.

Then $CR = AP$, for the triangles AMP and CMR are congruent. And since EG is equally inclined to AB and CD, the triangle APQ is isosceles. Therefore

$$CR = AQ; \quad \text{and} \quad QE = GR,$$

$$\therefore\ AQ + CR = AE + CG,$$

$$\therefore\ AQ = \tfrac{1}{2}(AE + CG).$$

Similarly

$$BX = \tfrac{1}{2}(BE + DG).$$

From the similarity of the triangles APQ, BXY we deduce

$$MR - MQ : NX - NZ = AE + CG : BE + DG.$$

Take on MN a point O such that

$$MO : ON = AE + CG : BE + DG,$$

and let S, T be the middle of QR and XZ. Then

$$MS = \frac{MR - MQ}{2}, \quad TN = \frac{NX - NZ}{2},$$

$$\therefore\ MS : TN = MO : ON, \quad \text{and} \quad MS \parallel NT.$$

Therefore S, O, T are collinear.

If KL be drawn through O parallel to EG, O is its middle, and KL is equally inclined to AB, CD. Hence O lies on the bisector of the angle made by these sides produced.

Again

$$MO : ON = AE + CG : BE + DG = AH + CF : BF + DH,$$

so that by drawing lines through M, O, N parallel to FG, we can similarly prove that O lies on the bisector of the angle included by AD, BC produced.

Consequently O is the centre of the inscribed circle.

Therefore it follows that

The middle points of the diagonals and the centre of the circle are collinear, and the lines drawn from O to M and N are proportional to $AE + CG$ and $BE + DG$.

By similarity we see also that

In a quadrilateral inscribed in a circle the bisectors of the angles, included by the two pairs of opposite sides produced, intersect on the join of the middle points of the diagonals, and the segments of the join thus divided stand in a simple proportion with the opposite sides divided in their points of intersection, and the point of concurrency is on the circle with the third diagonal of the quadrilateral as diameter.

S. IWATA'S THEOREM PROVED AND EXTENDED.

In the first number of the *Journal of the Mathematical Company in Tokyo* the following problem is proposed by S. Iwata:

Now, as in the figure[1]*, two oblique lines contain an ellipse, and four circles A, B, C, D are inscribed. Given the diameters of the circles A, B and C, it is required how to calculate the diameter of the circle D.*

Answer. Put down the diameter of the circle B, multiply by the diameter of the circle C, divide by the diameter of the circle A, and we get the diameter of the circle D as required.

In the same place it is added by S. Iwata himself that the solution of the problem was begun in the eighth month in 1864, and after so many attempts and after so many failures, it was first completed in the fifth month of 1866.

The above is given in the form of a problem, but it is in actuality nothing but a theorem, that may be enunciated in these words:

If there are four circles touching an ellipse and its two tangents, the product of the diameters of those that touch the ellipse externally is equal to the corresponding product of the internally touching ones; or in other words, the diameters of the four circles are in a simple proportion.

This theorem underwent the good luck of calling much attention on it in later years, when several researches were made on the subject, of which we here reproduce the following papers.

I.

H. Terao's Study.[2]

Iwata's theorem above given will be easy of solution, when the two tangents have the same directions as the axes of the ellipse. It was this specical case that was chiefly studied by S Iwata, who thereby encountered an immense mass of troubles and hardships, and who then generalised the result thus arrived at to the general case of the problem. Iwata only succeeds at his goal through the establishments of 55 relations. But the author of the present paper has come to generalise the proposition thus put forward by Iwata. Thus:

1) We omit the figure in this place.

2) The Journal of the Mathematico-Physical Society in Tokyo, Vol. 1, 1885.

Within an angle an ellipse or a hyperbola is inscribed, and other four ellipses or hyperbolas that are similar in form and position to a certain fixed conic are described touching the sides of the angle and also the first spoken conic, then the major or minor axes of these four conics will be in a simple proportion.

The four circles in Iwata's problem being all similar in shape and position to one another, it evidently forms a mere special case of our generalised proposition.

Before we enter into the establishment of our proposition, we shall a little tarry on the general treatment of conics that touch two straight lines and with each other. The application of such a preliminary consideration will leave our theorem exceedingly easy of demonstration.

The sides of the angle xOy being taken as axes of coordinates, the equations

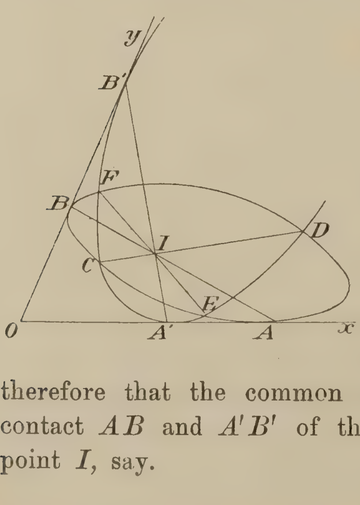
$$(s) \qquad k^2\left(\frac{x}{a}+\frac{y}{b}-1\right)^2-4xy=0,$$

$$(s') \qquad k'^2\left(\frac{x}{a'}+\frac{y^2}{b'}-1\right)^2-4xy=0$$

represent two conics that touch these axes at the distances a, b; a', b' respectively from the origin. And the equation

$$(s)-(s') \equiv k^2\left(\frac{x}{a}+\frac{y}{b}-1\right)^2-k'^2\left(\frac{x}{a'}+\frac{y}{b'}-1\right)^2=0$$

represents a conic that passes through the intersections of (s) and (s'). But the left-hand member being capable to be broken up into two linear factors, the conic represented by this equation consists of two straight lines, namely, two of the common chords of the two conics.



These two lines pass, as will be seen from their equation, through the intersection of the lines

$$\frac{x}{a}+\frac{y}{b}-1=0 \text{ and } \frac{x}{a'}+\frac{y}{b'}-1=0,$$

which are the chords of contact of (s) and (s') with the given lines. It appears therefore that the common chords CE and EF and the chords of contact AB and $A'B'$ of the two conics are all concurrent in the point I, say.

If now the values of k, k', a, a', b, b' undergo a change, the curves (s) and (s') will also change and will come in contact with each other in one occasion. This will happen when the ends of the chords CD and AB become coincident or when the two ends of one of these chords become so. The former will take place for $a = a'$, or $b = b'$; a case that has nothing to do with Iwata's problem.

In the latter case the chord whose two ends become coincident will become a tangent. In such a case we shall have a relation that exists between the constants.

We first take the common chord, whose equation is

$$\left(\frac{k}{a} + \frac{k'}{a'}\right)x + \left(\frac{k}{b} + \frac{k'}{b'}\right)y = k + k',$$

and try to find such a relation.

The value of y got from this equation being substituted in the equation of the curve (s), we obtain as the condition for the equality of its two roots the relation

$$k^2k'^2\left(\frac{k}{a} + \frac{k'}{a'}\right)\left(\frac{1}{b} - \frac{1}{b'}\right)^2 + k^2k'^2(k + k')\left(\frac{1}{b} - \frac{1}{b'}\right)\left(\frac{1}{ab'} - \frac{1}{a'b}\right)$$
$$- (k + k')\left(\frac{k}{b} + \frac{k'}{b'}\right) = 0$$

or

$$\text{(A)} \qquad k^2k'^2\left(\frac{1}{a} - \frac{1}{a'}\right)\left(\frac{1}{b} - \frac{1}{b'}\right) - (k + k')^2 = 0.$$

We have similarly for the second chord

$$\text{(B)} \qquad k^2k'^2\left(\frac{1}{a} - \frac{1}{a'}\right)\left(\frac{1}{b} - \frac{1}{b'}\right) - (k - k')^2 = 0.$$

We shall consider the curve (s) as a fixed curve and designate the curves that satisfy the relations (A) and (B) as *the curves of the classes or geneses* (A) *and* (B) respectively.

In forming the equation (A) we have assumed that $\frac{k}{b} + \frac{k'}{b'}$ is not equal to zero. If however this quantity be equal to zero, and at the same time $\frac{k}{a} + \frac{k'}{a'} \neq 0$, then the same equation (A) will be obtained by eliminating x between the equations of the straight line and of the curve and expressing the doubleness of the value of y.

In case when the relations

$$\frac{k}{a} + \frac{k'}{a'} = 0, \quad \frac{k}{b} + \frac{k'}{b'} = 0$$

simultaneously hold, there will give no curve of the class (A); and if

$$\frac{k}{a} - \frac{k'}{a} = 0, \quad \frac{k}{b} - \frac{k'}{b'} = 0,$$

there will give no curve of the class (B). In one of these cases the curves (s) and (s') are in similitude in respect to shape and position.

We employ the words *internal and external contacts* according as the centres of two ellipses or of two hyperbolas lie both on the same side of the common tangent or on its opposite sides. In the case of an ellipse and a hyperbola, these curves are called to have an *internal contact* when their common tangent lies between their centres, and an *external contact* in the contrary case.

Here we proceed to prove that the curves of the class (A) touch the curve (s) all in the same manner, and so also with the curves of the class (B).

When the common tangent of (s) and (s'), whose centres are (α, β) and (α', β'), has the equation $g(x, y) = 0$, the values of $g(\alpha, \beta)$ and $g(\alpha', \beta')$, will have the same or opposite signs according as the centres are on the same side of the tangent or otherwise.

For (α, β) the equations

$$\frac{k^2}{a}\left(\frac{\alpha}{a} + \frac{\beta}{b} - 1\right) - 2\beta = 0, \quad \frac{k^2}{b}\left(\frac{\alpha}{a} + \frac{\beta}{b} - 1\right) - 2\alpha = 0$$

will hold, whence we have

$$\alpha = \frac{k^2 a}{2(k^2 - ab)}, \quad \beta = \frac{k^2 b}{2(k^2 - ab)};$$

and similarly

$$\alpha' = \frac{k'^2 a'^2}{2(k'^2 - a'b')}, \quad \beta' = \frac{k'^2 b'}{2(k'^2 - a'b')}.$$

The tangent common to (s) and one of the class (A) will have the equation

$$g(x, y) = \left(\frac{k}{a} + \frac{k'}{a'}\right)x + \left(\frac{k}{b} + \frac{k'}{b'}\right)y - (k + k') = 0,$$

where the constants are connected by the equation (A). When (α, β) and (α', β') are substituted in the left-hand side of this equation, we obtain

$$g(\alpha, \beta) = -\frac{ab}{k^2 - ab}\left\{\frac{k^2k'^2}{ab}\left[1 - \frac{1}{2}\left(\frac{a}{a'} + \frac{b}{b'}\right)\right] - (k + k')\right\},$$

$$g(\alpha', \beta') = -\frac{a'b'}{k'^2 - a'b'}\left\{\frac{k'^2k^2}{a'b'}\left[1 - \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right] - (k + k')\right\}.$$

But the equation may be written

$$\frac{k^2k'^2}{ab}\left[1 - \frac{1}{2}\left(\frac{a}{a'} + \frac{b}{b'}\right)\right] + \frac{k^2k'^2}{a'b'}\left[1 - \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right] - k'(k + k') - k(k + k') = 0,$$

so that we have

$$\frac{g(\alpha, \beta)}{g(\alpha', \beta')} = -\frac{1}{k'}\left(\frac{k'^2}{a'b'} - 1\right) : \frac{1}{k}\left(\frac{k^2}{ab} - 1\right).$$

Similarly we have for a curve of the class (B)

$$\frac{g(\alpha, \beta)}{g(\alpha', \beta')} = + \frac{1}{k'}\left(\frac{k'^2}{a'b'} - 1\right) : \frac{1}{k}\left(\frac{k^2}{ab} - 1\right).$$

But (s) is an ellipse or a hyperbola according as $\frac{k^2}{ab} - 1$ is positive or negative, so also (s') according to the sign of $\frac{k'^2}{a'b'} - 1$.

As to the signs of k and k' nothing has been restrained, so that we may suppose their signs to be always positive. In this case the sign of $\frac{g(\alpha, \beta)}{g(\alpha', \beta')}$ depends only on those of $\frac{k^2}{ab} - 1$ and $\frac{k'^2}{a'b'} - 1$. Hence when (s) and (s') are both ellipses or both hyperbolas, the tangent common to (s) and one of the class (A) lies between their centres and these curves touch internally. If one of the curves is an ellipse and the other a hyperbola, their centres lie on the same side of the common tangent, so that they have an external contact.

A similar reasoning will reveal that the curves of the (B) class have always external contact with (s).

What we have said above, of course, does not apply when (s) or (s') is a parabola, a case that we exclude from our consideration.

Next, in order that (s') may be similar in form and position to a given conic (s_0), whose equation is

$$k_0^2\left(\frac{x}{a_0} + \frac{y}{b_0} - 1\right)^2 - 4xy = 0,$$

we shall have the conditions

$$\frac{k'^2}{a'^2} : \frac{k_0^2}{a_0^2} = \left(\frac{k'^2}{a'b'} - 2\right) : \left(\frac{k_0^2}{a_0 b_0} - 2\right) = \frac{k'^2}{b'^2} : \frac{k_0^2}{b_0^2},$$

whence we obtain

$$\frac{a'}{a_0} = \frac{b'}{b_0} = \lambda, \text{ say,}$$

so that $a' = \lambda a_0$, $b' = \lambda b_0$; and also $k' = \lambda k_0$.

These values render the equation of (s') to the form

$$k_0^2\left(\frac{x}{a_0} + \frac{y}{b_0} - \lambda\right)^2 - 4xy = 0.$$

This equation contains only one parameter λ, which can accordingly be determined by the relations (A) or (B). In other words, when (s') is made to touch (s), the curve will be completely determined in form and position.

It is to be noticed that the ratios, which the two axes A, B of (s') bear to the axes A_0, B_0 of (s_0), have each the value λ. For, the axes are respectively parallel, and so we have A and A_0 as like functions of k', a', b', θ and of k_0, a_0, b_0, θ respectively, where θ denotes the angle between the axes of coordinates:

$$A = F(k', a', b', \theta), \quad A_0 = F(k_0, a_0, b_0, \theta).$$

But the expression for A is to be linear and homogeneous in k', a', b', so that

$$F(\lambda k_0, \lambda a_0, \lambda b_0, \theta) = \lambda F(k_0, a_0, b_0, \theta),$$

or $A = \lambda A_0$, and similarly $B = \lambda B_0$.

Now the equations (A) and (B) may be written

$$k^2 k_0^2 \left(\frac{\lambda}{a} - \frac{1}{a_0}\right)\left(\frac{\lambda}{b_0} - \frac{1}{b}\right) - (\lambda k_0 \pm k)^2 = 0,$$

or

$$k_0^2 \left(\frac{k^2}{ab} - 1\right)\lambda^2 - kk_0\left[kk_0\left(\frac{1}{ab_0} + \frac{1}{ba_0}\right) \pm 2\right]\lambda + k^2\left(\frac{k_0^2}{a_0 b_0} - 1\right)^2 = 0,$$

which are both quadratic in λ, and each gives two values of λ, and with them two curves of each class.

There are therefore four conics that touch the curve (s) and its two tangents.

Since the above equations have the same coefficients for λ^2 and the constant terms are also equal, we infer that the products of their roots λ_1, λ_2 and of λ_3, λ_4 are equal to each other, that is, $\lambda_1\lambda_2 = \lambda_3\lambda_4$. But if the like axes of the four curves are denoted by A_1, A_2 and A_3, A_4, we shall have

$$\frac{A_1}{\lambda_1} = \frac{A_2}{\lambda_2} = \frac{A_3}{\lambda_3} = \frac{A_4}{\lambda_4} = A_0,$$

so that

$$A_1 A_2 = A_3 A_4, \text{ or } A_1 : A_3 = A_2 : A_4.$$

This last relation expresses the required extension of Iwata's theorem.

II.

J. Mizuhara's Result.[1])

H. Terao's extension of the theorem that is due to S. Iwata, as given above, seems to be more proper if we refer to the four conics as touching the *sides or sides produced* of the angle, for they do not necessarily happen to touch the sides only, as we have to point out below.

1) The Journal of the Mathematico-Physical Society in Tokyo, Vol. 4, pp. 267—275.

Moreover Terao's extended proposition is susceptible to be generalised in one way.

We retain Terao's notations in the following lines.

If an indeterminate conic

$$(s') \qquad k'^2\left(\frac{x}{a'}+\frac{y}{b'}-1\right)^2-4xy=0$$

be similar and similarly situated to a given conic

$$(s_0) \qquad k_0{}^2\left(\frac{x}{a_0}+\frac{x}{b_0}-1\right)^2-4xy=0,$$

and if (s') touch to the conic

$$(s) \qquad k^2\left(\frac{x}{a}+\frac{y}{b}-1\right)^2-4xy=0,$$

then the ratio λ of the corresponding axes of (s') and (s_0) is given by

$$k_0{}^2\left(\frac{k^2}{ab}-1\right)\lambda^2-kk_0\left[kk_0\left(\frac{1}{ab_0}+\frac{1}{ba_0}\right)\pm 2\right]\lambda+k^2\left(\frac{k_0{}^2}{a_0b_0}-1\right)=0,$$

which we designate as (A) and (B).

In like manner, if the undetermined conic (T')

$$(\mathrm{T}') \qquad k'^2\left(\frac{x}{b'}+\frac{y}{a'}-1\right)^2-4xy=0,$$

which is similar and similarly situated to a fixed conic

$$(\mathrm{T}_0) \qquad k_0{}^2\left(\frac{x}{b_0}+\frac{y}{a_0}-1\right)^2-4xy=0,$$

touch to (s), then the ratio λ of the corresponding axes of (T') and (T_0) will be given by

$$k_0{}^2\left(\frac{k^2}{ab}-1\right)\lambda^2-kk_0\left[kk_0\left(\frac{1}{aa_0}+\frac{1}{bb_0}\right)\pm 2\right]\lambda+k^2\left(\frac{k_0{}^2}{b_0a_0}-1\right)=0,$$

which we call (A') and (B').

These four equations in λ coincide in the coefficients of λ^2 and in the constant terms. Hence if the roots of these equations be denoted by λ_1 and λ_2, λ_3 and λ_4, λ_1' and λ_2', λ_3' and λ_4', we have

$$\lambda_1\lambda_2=\lambda_3\lambda_4=\lambda_1'\lambda_2'=\lambda_3'\lambda_4'.$$

But the two conics (s_0) and (T_0) are equal in every respect, and their corresponding axes are equally inclined to the bisector of the angle between the coordinate axes, for x and y being interchanged in the equation (T) we obtain the equation (s_0). Hence the corresponding axes of these conics are evidently equal. Let R_0 be one of their axes. Corresponding axes of the curves belonging to the classes

(A), (B), (A′) and (B′) being denoted by R_1, R_2; R_3, R_4; R_1', R_2'; R_3', R_4', we have

$$R_1 R_2 = \lambda_1 \lambda_2 R_0^2, \qquad R_3 R_4 = \lambda_3 \lambda_4 R_0^2,$$
$$R_1' R_2' = \lambda_1' \lambda_2' R_0^2, \quad R_3' R_4' = \lambda_3' \lambda_4' R_0^2,$$

whence we get

$$R_1 R_2 = R_3 R_4 = R_1' R_2' = R_3' R_4'.$$

It follows therefore the proposition:

If an ellipse or a hyperbola be described touching the sides of an angle, and also eight conics, similar in form to a given ellipse or hyperbola, be described touching the sides or sides produced of the same angle, and if the corresponding axes of these conics be equally inclined to the bisector of the angle, then the products of the corresponding axes of the pairs of each two conics similar in positions and touching externally or internally to the former ellipse or hyperbola are all equal.

We next consider the limits of λ and k_0.

From the formulae (A) and (B) we obtain

$$\pm k_0 = \frac{k\lambda \pm k^2 \sqrt{\dfrac{1}{ab}\lambda^2 - \left(\dfrac{1}{ab_0} + \dfrac{1}{ba_0}\right)\lambda + \dfrac{1}{a_0 b_0}}}{\left(\dfrac{k^2}{ab} - 1\right)\lambda^2 - k^2\left(\dfrac{1}{ab_0} + \dfrac{1}{ba_0}\right)\lambda + \dfrac{k^2}{a_0 b_0}}.$$

The quantity under the radical sign must be positive. This being denoted by ε^2, we have

$$\lambda = \frac{1}{2}\left(\frac{b}{b_0} + \frac{a}{a_0}\right) \pm \frac{1}{2}\sqrt{\left(\frac{b}{b_0} - \frac{a}{a_0}\right)^2 + 4ab\varepsilon^2}.$$

Now a and b being taken both for positive, λ cannot lie between $\dfrac{b}{b_0}$ and $\dfrac{a}{a_0}$. In other words, the chords of contact (in respect to the sides of the angle) of (s') and (s) cannot intersect between their points of contact (these points not included).

From (A) and (B) follows

$$\lambda = \frac{1}{2k_0\left(\dfrac{k^2}{ab} - 1\right)}\left\{\left[kk_0\left(\frac{1}{ab_0} + \frac{1}{ba_0}\right) \pm 2\right]\right.$$
$$\left. \pm k\sqrt{k_0^2\left[k^2\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2 + \frac{4}{a_0 b_0}\right] \pm 4kk_0\left(\frac{1}{ab_0} + \frac{1}{ba_0}\right) + \frac{4k^2}{ab}}\right\},$$

of which the quantity under the radical sign must be positive, equal to ε_1^2, say. Hence we have

$$\mp k_0 = \frac{1}{k^2\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2 + \frac{4}{a_0b_0}} \left\{2k\left(\frac{1}{ab_0} + \frac{1}{ba_0}\right)\right.$$

$$\left. \pm \sqrt{4k^2\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2\left(1 - \frac{k^2}{ab}\right) + \varepsilon_1{}^2\left[k^2\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2 + \frac{4}{a_0b_0}\right]}\right\}.$$

Assume a_0 and b_0 to be positive (and the same assumption will be taken throughout in the following). Then

$$k^2\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2 + \frac{4}{a_0b_0} > 0,$$

so that, if $1 - \frac{k^2}{ab} > 0$, that is, if (s) is a hyperbola, $\mp k_0$ cannot lie between

$$\frac{-\left[k^2\left(\frac{1}{a^2b_0{}^2} - \frac{1}{b^2a_0{}^2}\right) \mp \frac{4}{a_0b_0}\right]\sqrt{1 - \frac{k^2}{ab}}}{\left[k\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2 + \frac{4}{a_0b_0}\right]^2},$$

that is, $\frac{k_0{}^2}{a_0b_0} - 1$ cannot lie between

$$\frac{-\left[k^2\left(\frac{1}{a^2b_0{}^2} - \frac{1}{b^2a_0{}^2}\right) \mp \frac{4}{a_0b_0}\sqrt{1 - \frac{k^2}{ab}}\right]^2}{\left[k\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2 + \frac{4}{a_0b_0}\right]^2},$$

which are both negative.

Consequently (s_0) may have any form when it is an ellipse.

If $1 - \frac{k^2}{ab} < 0$, or if (s) is an ellipse, there is no limits to the values of $\mp k_0$.

And similarly for the curves of (A′) and (B′).

These results being briefly summarised, for the existence of the eight conics we see that

(1) *The chord of contact of (s) cannot intersect with the chords of contact of the eight conics between its points of contact.*

(2) *When (s) and (s_0) are both hyperbolas, k_0 cannot lie between*

$$\frac{\pm 2k\left(\frac{1}{ab_0} + \frac{1}{ba_0}\right) \mp 2k\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)\sqrt{1 - \frac{k^2}{ab}}}{k^2\left(\frac{1}{ab_0} - \frac{1}{ba_0}\right)^2 + \frac{4}{a_0b_0}}$$

and the quantities obtained from these by reversing the second double signs, and cannot also lie between the quantities the same as before except that a_0 and b_0 should be interchanged in them.

(3) *When (s) is a hyperbola and (s_0) an ellipse, the latter may be of any form.*

(4) *When (s) is an ellipse, (s_0), whether it be an ellipse or a hyperbola, may be of any form whatever.*

Next something on the sign of λ. As a_0 and b_0 are positive, (s') and (T') will have contacts with the sides, if λ be positive. Here we shall only speak of the curves of the classes (A) and (B); for the results for (A′) and (B′) can be deduced by the interchange of a_0 and b_0.

(1) *If* (s) *and* (s_0) *are both ellipses*, then

$$\frac{k_0^2}{a_0 b_0} - 1 > -\frac{4a^2b^2a_0b_0\left(\frac{k^2}{ab}-1\right)+k^2(ba_0-ab_0)^2}{k^2(ba_0+ab_0)},$$

that is,

$$\frac{k_0^2}{a_0 b_0} - 1 > \frac{4}{a_0b_0k^2\left(\frac{1}{ab_0}+\frac{1}{ba_0}\right)} - 1,$$

or

$$k^2k_0^2\left(\frac{1}{ab_0}+\frac{1}{ba_0}\right)^2 > 4,$$

so that the four values of λ that arise from (A) and (B) are all positive, and so *the four conics all touch the sides of the angle.*

(2) *If* (s) *and* (s_0) *are both hyperbolas*, the two roots of λ in (A) are both negative for the same sign of k and k', and those in (B) are both negative or both positive according as

$$k^2k_0^2\left(\frac{1}{ab_0}+\frac{1}{ba_0}\right)^2 >< 4.$$

When the signs of k and k_0 are different, we have only to interchange (A) with (B). Therefore *the four conics all touch the sides produced, or two curves touch the sides and the remaining two the sides produced*, according to the conditions expressed by the above inequalities.

(3) *If* (s) *and* (s_0) *are the one an ellipse and the other a hyperbola*, the signs of the two roots in (A) as well as in (B) are both different. In this case *two of the curves touch the sides and the others touch the sides produced.*

As to the points of contact of (s) *with* (s') *or* (T'), *they lie on the sides, when* (s) *is an ellipse.*

When (s) *is a hyperbola, its points of contact with two curves of the same class are the one on the sides and the other on their prolongations.* This will be evident if we notice that one of the branches of the hyperbola lies within the angle and the other in the opposite angle, and if we take notice on the

Theorem. The straight line that joins the points of contact of (s) *with two curves of the same class passes necessarily through the centre of* (s). *(In this theorem the straight line may be replaced by a conic.)*

To prove this theorem, we take the equations of the tangents common to (s) and two curves of the class (A),

$$\left(\frac{k}{a}+\frac{k_0}{a_0}\right)x+\left(\frac{k}{b}+\frac{k_0}{b_0}\right)y-(k+\lambda k_0)=0,$$

where λ is to be replaced by λ_1 and λ_2. These two lines are evidently parallel. Hence the chord of contact is a diameter of (s), and therefore passes through its centre. The same can be said with curves of other classes. We have therefore our proposition proved.

If we allow that (s), (s') or (T_0) may touch not only the sides, but also the sides produced, then what we have said in the above about the limits of λ and k_0 and the sign of λ will introduce more cases in addition to those already considered.

III.

T. Hayashi's further extension.[1])

A.

1. Taking to axes the common tangents to two given conics, these may be represented by

$$S\equiv k^2L-4xy=0,\quad S'\equiv k'^2L^2-4xy=0,$$

whose common chord of contact is

$$L\equiv ax+by+1=0.$$

A conic that has a double contact with (S) is

$$A\equiv k^2L^2-4xy+\mu^2M^2=0$$

with

$$M\equiv \alpha x+\beta y+1=0$$

for the common chord of contact.

$$S'-A\equiv (k'^2-k^2)L^2-\mu^2M^2=0,$$

which is the conic passing through the intersections of (A) and (S'), consists evidently of two straight lines $\lambda L\pm\mu M=0$, that pass through the intersection of (L) and (M).

Taking the upper sign, for the condition that it touches (S') and consequently touches also (A) we obtain

$$k'^2\mu^2(a-\alpha)(1,-\beta)-(\lambda+\mu)^2=0.$$

1) The Journal of the Mathem.-Phys. Soc. in Tokyo, Vol. VI, pp. 41—51, April, 1895.

2. In the conic

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

whose centre is (ξ, η), we change the axes according to the formulae

$$x = \xi + lX + mY, \quad y = \eta + l'X + m'Y,$$

when we get, $\varDelta$ representing the discriminant,

$$(Al^2 + 2Hll' + Bl'^2)x^2 + (Am^2 + 2Hmm' + Bm'^2)y^2 + \frac{\varDelta}{AB - H^2} = 0,$$

for it should be

$$A\xi + H\eta + G = 0, \quad H\xi + B\eta + F = 0,$$

$$Alm + 2H(lm' + l'm) + Bl'm' = 0.$$

We have thus for the semi-axes

$$[\bar{a}]^2 = \frac{\varDelta}{(H^2 - AB)(Al^2 + 2Hll' + Bl'^2)},$$

$$[\bar{b}]^2 = \frac{\varDelta}{(H^2 - AB)(Am^2 + 2Hmm' + Bm'^2)}.$$

3. If (A) be similar and similarly situated to the conic treated in last article, we shall have

$$\frac{A}{k^2a^2 + \mu^2\alpha^2} = \frac{H}{k^2ab - 2 + \mu^2\alpha\beta} = \frac{B}{k^2b^2 + \mu^2\beta^2} = \gamma, \quad \text{say.}$$

Hence the coefficients in (A) being denoted by $A_1, B_1, \ldots$, we have for semi-major axis the relation

$$\frac{[a]^2}{[\bar{a}]^2} = \frac{\varDelta_1(H^2 - AB)(Al + 2Hll' + Bl'^2)}{\varDelta(H_1^2 - A_1B_1)(A_1l_1 + 2H_1l_1l_1' + B_1l_1'^2)} = \gamma^3\frac{\varDelta_1}{\varDelta}.$$

Now on calculation we have

$$\varDelta_1 = 4\{k^2\mu^2(a - \alpha)(b - \beta) - (k^2 + \mu^2)\}$$

$$= \frac{4}{k'^2}\{k^2(\lambda + \mu)^2 - k'^2(k^2 + \mu^2)\} = -\frac{4}{k'^2}(\lambda\mu - \lambda^2)^2$$

by the condition given in the end of article 1.

$$\therefore\ [a]^2 = K[\lambda\mu - k^2], \quad \text{where} \quad K^2 = -\frac{4\gamma^2}{k'^2\varDelta}[\bar{a}]^2.$$

4. From the relation in last article we have

$$\left(\frac{A}{\gamma} - k^2a^2\right)\left(\frac{B}{\gamma} - k^2b^2\right) - \left(\frac{H}{\gamma} - k^2ab + 2\right)^2 = 0,$$

that gives us two values of γ, the one of which we need only to select, since the other corresponds to the conic touching (S) on the opposite side of (S'). Hence

$$\mu\alpha = \pm\sqrt{\frac{A}{\gamma} - k^2a^2}, \quad \mu\beta = \pm\sqrt{\frac{B}{\gamma} - k^2b^2}.$$

Of these it may be shown that there are two sets of values with opposite signs. μ_1 and μ_2 being the two values of μ corresponding to the one set, we have for product

$$\mu_1 \mu_2 = \frac{k'^2\mu^2\alpha\beta - \lambda^2}{k'^2 ab - 1} = \frac{k'^2\left(\frac{H}{\gamma} - k^2 ab + 2\right) - \lambda^2}{k'^2 ab - 1},$$

which is equal to a constant. The product of the other values of μ, say μ_3 and μ_4, will also have the same value, thus $\mu_1\mu_2 = \mu_3\mu_4$.

We therefore have

$$\left(\frac{[a_1]}{K} + k^2\right)\left(\frac{[a_2]}{K} + k^2\right) = \left(\frac{[a_3]}{K} + k^2\right)\left(\frac{[a_4]}{K} + k^2\right),$$

or

$$[a_1][a_2] - [a_3][a_4] = k^2 K\{[a_3] + [a_4] - [a_1] - [a_2]\},$$

which, specially for $K = 0$, is the very result obtained by H. Terao.

Now let us consider the case $k' = 0$, excluded in the above. In this case $S' \equiv xy = 0$, and the condition of the conic $(S') - (A)$ touching with (S') becomes $a = \alpha$, or $b = -\beta$, so that the conic touches the x- or y-axis at the point of contact of (S) with the same axis.

$$\therefore\ \Delta_1 = -4(k^2 + \mu^2) = -4\frac{A}{a^2\gamma}, \quad \text{or} \quad = -4\frac{B}{b^2\gamma},$$

by the first formula in article 3.

Hence writing $K' = -4\gamma^2[a]^2/\Delta$, we get

$$[a_1]^2 = K'\frac{A}{a^2} \quad \text{and} \quad [a_2]^2 = K'\frac{B}{b^2}, \quad \therefore\ \frac{[a_1]^2}{[a_2]^2} = \frac{A}{a^2} : \frac{B}{b^2}.$$

In particular if the conic in article 2 be a circle, then $A = B$, and we obtain the following result:

Two circles are each in double contact with a given conic touching two given conics at A and B. If the two tangents meet at O, then the ratio of the radii of the two circles is equal to that of OA to OB.

The ratio of the radii of curvature at A and B is equal to $OA^3 : OB^3$, as is well known. Therefore

The ratio of the cubes of the radii of two circles each having a double contact with a given ellipse or hyperbola is equal to the ratio of the radii of curvature at the point of contact.

Again the radius of curvature of a central conic at a point is proportional to the cube of the semi-diameter conjugate to the diameter through that point; and the same is also true for the other circles corresponding to other value of γ. Therefore

The ratio of the radii of two circles each having a double contact with a given ellipse or hyperbola is equal to the ratio of the semi-diameters conjugate to the diameters through the points of contact.

The ratio of the radii of two circles in double contact with a given ellipse or hyperbola (one circumscribing and the other inscribing) having one common point of contact is constant.

B. Extension to the case in space.

Let the right circular cone $yz + zx + xy = 0$ be inscribed with the quadrics

$$S = k^2\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1\right)^2 - 4(yz + zx + xy) = 0,$$

$$S' = k'^2\left(\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} - 1\right)^2 - 4(yz + zx + xy) = 0,$$

where a, b, c; a', b', c' denote the distances along the axes from the origin to the points of contact. From these by subtraction we get

$$S - S' = k^2\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1\right)^2 - k'^2\left(\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} - 1\right)^2 = 0,$$

which evidently represents two planes.

If (S) and (S') touch each other, one of these planes will be the tangent plane at the point of contact. To find this condition we form the equation to the tangent plane of (S') at (x', y', z') and compare it with one of the two factors in above equation, when we obtain after reduction

$$k\left(\frac{1}{a'} - \frac{1}{a}\right)\left(\frac{x'}{a'} + \frac{y'}{b'} + \frac{z'}{c'} - 1\right) - 2\,\frac{k + k'}{k'^2}\,(y' + z') = 0,$$

with two like formulae obtained by cyclically changing a, x; b, y; c, z.

From these we have

$$\frac{x'}{k\left\{\left(\frac{1}{b'} + \frac{1}{c'} - \frac{1}{a'}\right) - \left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\right\}} = \text{etc.} = \frac{\frac{x'}{a'} + \frac{y'}{b'} + \frac{z'}{c'} - 1}{4\,\frac{k + k'}{k'^2}}$$

$$\therefore\ = \frac{1}{k\left[\sum \frac{1}{a'}\left(\frac{1}{b'} + \frac{1}{c'} - \frac{1}{a'}\right) - \sum \frac{1}{a'}\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\right] - 4\,\frac{k + k'}{k'^2}}$$

$$= -\frac{\sum \frac{x'}{a} - 1}{4\,\frac{k + k'}{kk'}} \quad \left(\text{since } k'\left(\sum \frac{x'}{a'} - 1\right) + k\sum\left(\frac{x'}{a} - 1\right) = 0\right)$$

$$= \frac{1}{k\left[\sum \frac{1}{a}\left(\frac{1}{b'} + \frac{1}{c'} - \frac{1}{a'}\right) - \sum \frac{1}{a}\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right) + 4\,\frac{k + k'}{kk'}\right]},$$

whence, after reduction, we get the condition in the form

$$\text{(A)}\quad k^2k'^2\left\{2\sum\left(\frac{1}{b}-\frac{1}{b'}\right)\left(\frac{1}{c}-\frac{1}{c'}\right)-\sum\left(\frac{1}{a}-\frac{1}{a'}\right)^2\right\}-4\,(k+k')^2=0,$$

and similarly for the other of the two planes in $(S)-(S')$,

$$\text{(B)}\quad k^2k'^2\left\{2\sum\left(\frac{1}{b}-\frac{1}{b'}\right)\left(\frac{1}{c}-\frac{1}{c'}\right)-\sum\left(\frac{1}{a}-\frac{1}{a'}\right)^2\right\}-4\,(k-k')^2=0.$$

Now let (S') be similar to (B) and similarly situated to

$$S_0\equiv k^2\left(\frac{x}{a_0}+\frac{y}{b_0}+\frac{z}{c_0}-1\right)^2-4\,(yz+zx+xy)=0,$$

then we have

$$\frac{\dfrac{k'^2}{a'^2}}{\dfrac{k_0^{\,2}}{a_0^{\,2}}}=\text{etc.}=\frac{\dfrac{k'^2}{b'c'}-2}{\dfrac{k_0^{\,2}}{b_0c_0}-2}=\text{etc.}$$

$$\therefore\quad \frac{a'}{a_0}=\frac{b'}{b_0}=\frac{c'}{c_0}=\frac{k'}{k_0}=\lambda,\ \text{say}.$$

Thus λ is the ratio of the parallel central radii of (S') and (S_0). Hence a', b', c', k' in (A) and (B) being substituted by $a_0\lambda, b_0\lambda, \ldots$, we see, from the resulting equations, that the products of their roots should be equal, or $\lambda_1\lambda_2=\lambda_3\lambda_4$.

But if r_1, r_2, r_3, r_4 denote the radii vectores of the four conics $(S_1), (S_2), (S_3), (S_4)$ parallel to the radius vector r_0 of (S_0), then

$$\lambda_1=\frac{r_1}{r_0},\quad \lambda_2=\frac{r_2}{r_0},\quad \ldots,$$

so that

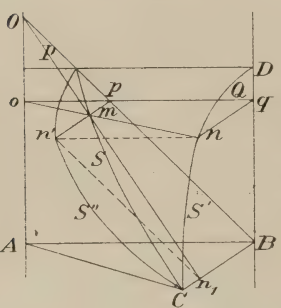
$$r_1r_2=r_3r_4\quad\text{or}\quad\frac{r_1}{r_3}=\frac{r_4}{r_2}.$$

Thus *the parallel central radii of four similar and similarly situated quadrics inscribed in a right circular cone and touching a given central quadric also inscribed in the cone form a simple proportion.*

H. TERAO, ON THE SECTIONS OF A WEDGE.[1]

A *wedge surface* is a surface, as is well known, that will be formed by the straight lines drawn parallel to a given plane and crossing a given straight line and besides a given curve or another given straight line. The given straight line is called the *axis* of the wedge and the given plane its *directrial plane.*

We here propose to give a proof to a simple relation that exists between the sections of a wedge made by any planes and one made by a plane parallel to the axis.



Draw from any point A on the axis of a wedge a plane parallel to the directrial plane. Through the intersection BC of this plane with any plane P, draw the plane Q parallel to the axis. Between the sections of the wedge made by the planes P and Q we find a simple relation which we are going to describe.

Let these sections be S and S'.

The plane Q remains indeterminate in position for a certain position of P, for the point A may have any position on the axis.

Draw a plane through the axis AO, cutting the two planes P and Q along BO and BD respectively. BD is of course parallel to OA.

To obtain points on the sections S and S', we draw a plane opn parallel to the directrial plane. This plane cuts the wedge along a generating line on and cuts the planes P and Q along the straight lines pm and qn, which intersect with the generatrix in m and n, points on the curves S and S'. But pm and BC are parallel, since they are the intersections of one and the same plane with two parallel planes. Similarly qn and BC are parallel. Consequently pm and qn are parallel. We have therefore

$$pm : qn = op : oq.$$

The straight lines oq and AB are parallel, because they are the intersections of a plane with two parallel planes. BD and AO are parallel as already mentioned. Hence

$$op : oq = Op : OB, \quad \text{and} \quad \therefore\ pm : qn = Op : OB.$$

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1) The Journal of the Math.-Phys. Soc. in Tokyo, Vol. 2, pp. 10—15, 1885.

Consequently if Bn_1 be taken on BC equal to qn, the point n_1 lies on the straight line Om. If the curve S be projected on the plane P parallel to AB, the projection n' of n is on the line pm, and pn' is equal to qn, and hence equal to Bn_1. Hence the two points n' and n_1 are on a straight line parallel to OB.

We therefore derive the following construction.

First describe a curve S'' which is the projection of S' on the plane P along lines parallel to AB. Next draw from the point O any straight line On_1 and from its intersection n_1 with CB draw n_1n' parallel to OB. And lastly draw from n', where this line cuts S'', the line $n'p$ parallel to BC. Then the point m, intersection of $n'p$ and On_1, is a point that belongs to the curve S.

Accordingly the description of the curve S can be easily effected, if we know how to describe the curve S''.

Here we give an example for illustration.

Describe a circle whose centre lies on a plane R, and whose plane is normal to it. The surface that will be formed by straight lines that cross this circle and a certain straight line, is a wedge surface of a special kind, namely a *circular wedge*. If this wedge be cut normal both to R and to the directrial plane, the section is always an ellipse. The projection S'' of this ellipse on a plane I normal to R is also an ellipse. Therefore the section S made by the plane P can be described according to the method above stated by means of an ellipse and straight lines. But one of the axes of this ellipse lies on the intersection of P and R and has a constant length, a say, while the other varies from 0 to infinity according to the positions of the point A, so that by suitably choosing the point A we can make that axis of this ellipse S'' equal to a. In this case the ellipse S'' becomes a circle. Hence the sections of a circular wedge made by planes normal to R can be described by means of a circle and straight lines.

If we take the point A on the middle of the axis of a circular wedge, the line BC will have the same direction as one of the axes of the ellipse S''. In this case the areas of the curves S and S'' are equal. (Hereby we assume the point O is not within the curve S''.)

This proposition will hardly need a proof, because it is too simple.

If the point O be within S'', the curve S will be divided by this point into two parts. In this case the area of the curve S'' will be equal to the difference of the areas of the two parts of S.

H. TERAO, ON THE CURVE WHOSE AREA IS EQUAL TO THE GEOMETRICAL MEAN OF THOSE OF TWO HOMOTHETIC CURVES.[1]

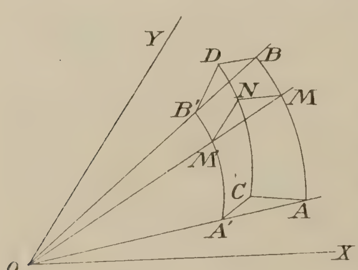
In the present paper we shall carry our study by the methods of elementary geometry.

First a few words on the terms *homothetic* and *homothety.*

If we take a point M' on the radius vector drawn from a fixed point O to any point M that lies on a given surface or curve such that $OM' : OM$ is constant, so the locus of M' will describe a surface or curve, which is called to be mutually homothetic with the former surface or curve. Hereby the points M and M' are termed a pair of corresponding points, and the constant ratio $OM' : OM$ the *ratio of homothety,* and the point O the *centre of homothety.*

Moreover, when OM and OM' are taken in the same sense, we say the homothety is positive, otherwise negative.

In this place we restrict ourselves to the case of positive homothety and the curves coplanar with the centre of homothety.



Theorem. From a pair of corresponding points M and M' on the arcs AB and $A'B'$ of two plane curves homothetic in respect to the centre O parallels are drawn in two invariable directions OX and OY to meet in N. The area of the sector COD, that has its base on the arc of the locus of N, will be the geometrical mean of the areas of the sectors AOB and $A'OB'$ of the two given curves AB and $A'B'$. (The radius vector is supposed to turn always in the same direction while the point M moves from A to B along the curve AB.)

The curve CD will be termed the *mean curve* of AB and $A'B'$.

We will for the first place prove the theorem for the special case where the curve AB is a straight line. In this case the curve $A'B'$ homothetic to AB is also evidently a straight line. The mean curve CD will be a straight line too.

1) The Journal of Phys. Sch. in Tokyo, Vol. 2, pp. 137–143, May, 1891. Revised from a paper previously published in the Journal of the Senkō Gaku-Sha.

For let the points where AB cuts OX and OY be E and F, and those that $A'B'$ cuts be G and H. Then the line drawn from M parallel to OX cuts the line FG in the ratio $FM : ME$; and the line drawn from M' parallel to OY cuts the same line in the ratio $HM' : M'G$. But these ratios are equal. Therefore the parallels from M and M' meet on the line FG, which is consequently the locus of N and is the mean curve of AB and $A'B'$.

Draw AP, BQ, $A'P'$, $B'Q'$ parallel to OY and Aa, Bb, $A'a$, $B'b$ normal to OX. Then the triangle OAB is, in area, equal to half the difference of the parallelograms $BQ \cdot OP$ and $B'Q' \cdot OP'$, as will be easily proved. Hence denoting the areas of the triangles OAB, $OA'B'$, OCD by S, S', Σ respectively, we have

$$S = \tfrac{1}{2}(OP \cdot Bb - OQ \cdot Aa),$$
$$S' = \tfrac{1}{2}(OP' \cdot B'b' - OQ' \cdot A'a'),$$
$$\Sigma = \tfrac{1}{2}(OP' \cdot Bb - OQ' \cdot Aa).$$

The ratios $\dfrac{OP'}{OP}$, $\dfrac{OQ'}{OQ}$, $\dfrac{A'a'}{Aa}$, $\dfrac{B'b'}{Bb}$ are all equal to the ratio of homothety, k say. Hence we have

$$S' = k^2 S, \quad \Sigma = kS,$$

whence the elimination of k gives $\Sigma^2 = SS'$, which establishes our theorem for the case of a straight line.

We next prove the theorem for the case when the homothetic curves are not straight lines.

In this case the mean curve is also a curved line.

The sectors on the homothetic curves as bases may be considered as composed of small triangles S_1, S_2, S_3, ... and S_1', S_2', S_3', ... respectively, that stand on linear bases which are mutually homothetic.

The sector of the mean curve will be decomposed into the sum of the triangles Σ_1, Σ_2, Σ_3, ... that have their bases which are the mean lines of the components of the two homothetic curves.

Thus from what we have said above we obtain for these triangles

$$S_1' = k^2 S_1, \quad S_2' = k^2 S_2, \ \ldots;$$

and

$$\Sigma_1 = kS_1, \quad \Sigma_2 = kS_2, \ \ldots$$

The sums of these quantities being denoted by S, S' and Σ, we have

$$S' = k^2 S, \quad \Sigma = kS, \quad \therefore \ \Sigma^2 = SS'.$$

The same relation will also apply for the case of any curves, for they may be looked upon as the limits when the number of small linear parts are indefinitely increased

Corollary 1. Our theorem will still hold for two closed homothetic curves that enclose the centre of homothety, the entire areas of the curves being referred to.

In the case when every radius vector cuts the curves each in only one point, the proposition will be easy to prove, for we can apply our theorem separately to the parts into which we consider the curves divided, and then sum up the results.

In the case when the radius vector cuts the curves in more than one points the same reasoning will apply, only that some of the areas are to be subtracted in place of being added.

Corollary 2. In the case when the centre of homothety lies outside the curves our proposition will still hold.

For, when the curves are met with by every radius vector in two points their areas will be equal each to the difference of the areas of two sectors that stand on arcs of the curves.

In other cases we may also treat in a like manner.

Our proposition will be therefore found correct in every case imaginable. It is therefore generally true.

Example 1. The mean figure of two homothetic squares A and B, their sides being taken for assigned directions, is a rectangle whose two sides are equal to the sides a, b of the squares. Hence the area of such a rectangle is equal to the geometrical mean of a^2 and b^2, that is, equal to ab.

Example 2. If the mean curve of two concentric circles whose radii are a and b will be formed in respect to two orthogonal diameters as invariable directions, this curve is an ellipse with the axes $2a$ and $2b$. Hence the area of such an ellipse is equal to the geometrical mean of πa^2 and πb^2, that is, equal to πab.

H. TERAO, ON THE SURFACE WHOSE VOLUME IS EQUAL TO THE GEOMETRICAL MEAN OF THOSE OF THREE HOMOTHETIC SURFACES.[1])

In this place we try to extend the result of the preceding paper to the case of surfaces. There the problem was attacked in an elementary treatment. Here we prefer however the analytical method for fear of intricating ourselves in useless entanglements.

1. *The volume V of the tetrahedron whose vertices are* (0, 0, 0), (x, y, z), (x', y', z') *and* (x'', y'', z'') *is proportional to the determinant*

$$\begin{vmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{vmatrix}.$$

2. *A surface homothetic to a plane in respect to a point O is a parallel plane.*

For, take O for the origin of coordinates and let the equation to the given plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

and let (x', y', z') be the point on the surface that corresponds to (x, y, z). Then the ratio of homothety being k, we have

$$\frac{x'}{x}=\frac{y'}{y}=\frac{z'}{z}=k,$$

and we obtain

$$\frac{x'}{a}+\frac{y'}{b}+\frac{z'}{c}=k.$$

3. *If from the corresponding points of three homothetic planes three planes are drawn respectively parallel to three given planes, no two of which are parallel, the locus of their point of intersection is also a plane.*

Take the centre of homothety for the origin and the directions of the three given planes for coordinate planes. Let the equation to one of the three homothetic planes be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

1) The Journal of Phys. Sch. in Tokyo, Vol. 2, pp. 201—205, July, 1891.

If then (x', y', z') and (x'', y'', z'') denote the coordinates of those points that correspond to (x, y, z), and k and k' the ratios of homothety, we shall have

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = k, \quad \frac{x''}{x} = \frac{y''}{y} = \frac{z''}{z} = k'.$$

Let (x_1, y_1, z_1) be the point of intersection of the planes drawn through (x, y, z), (x', y', z') and (x'', y'', z'') parallel to the planes yz, zx, xy, respectively. We have then evidently $x_1 = x$, $y_1 = y'$, $z_1 = z''$; that is,

$$x_1 = x, \quad y_1 = ky, \quad z_1 = k'z,$$

values that transform the preceding equation into

$$\frac{x_1}{a} + \frac{y_1}{kb} + \frac{z_1}{k'c} = 1.$$

4. *If we replace the three homothetic planes by three homothetic straight lines, the resulting locus will be a straight line.* For we can draw three homothetic planes through the three lines, and the locus before us becomes the intersection of two planes.

Therefore *the locus of intersection of three planes respectively parallel to three given planes drawn through the corresponding points of three homothetic polygons is also a polygon.*

5. *Let $A_1B_1C_1$ be the locus of the intersection of the planes drawn respectively parallel to three planes P, Q, R, no two of which are parallel, from three corresponding points in three homothetic triangles ABC, $A'B'C'$, $A''B''C''$. Then the volume of the tetrahedron with the vertex at the centre of homothety and with $A_1B_1C_1$ as base is equal to the geometrical mean of the three tetrahedrons on ABC, $A'B'C'$ and $A''B''C''$ as bases.*

For take the origin at the centre of homothety and the axial planes parallel to P, Q, R. Then by Article 1 the volumes V, V', V'', V_1 of the tetrahedrons $OABC$, $OA'B'C'$, $OA''B''C''$, $OA_1B_1C_1$, are

$$V = AD, \quad V' = AD', \quad V'' = AD'', \quad V_1 = AD_1,$$

where A depends only on the mutual positions of the planes P, Q, R, while the D's are determinants of the third order.

Since the coordinates of A', B', C' and A'', B'', C'' are respectively k times and k' times of those of A, B, C, we have, according to the property of determinants, $D' = k^3 D$, $D'' = k'^3 D$.

The coordinates of A_1, B_1, C_1 are equal to those of A, B', C'', so that the first vertical line of the determinant D_1 is equal to that of D, the second to k times that of D and the third to k' times that of D. Consequently $D_1 = kk'D$.

Hence we have

$$V' = k^3 V, \quad V'' = k'^3 V, \quad V_1 = kk' V,$$

whence follows

$$V V' V'' = V_1^3, \text{ or } V_1 = \sqrt[3]{V V' V''},$$

as was to be shown.

6. The volumes of the conical solids, that have to bases three homothetic surfaces and their vertices at the centre of homothety, may be conceived to consist of an indefinitely great number of small tetrahedrons with three homothetic small triangular bases. It follows therefore from last proposition that the volume of the cone whose base is the locus of the point of intersection of three planes drawn through corresponding points of three homothetic surfaces respectively parallel to three given planes is equal to the geometrical mean of the volumes of the cones on the three homothetic surfaces as bases.

Our proposition still holds for closed surfaces.

We call such a fourth surface as just described the *mean surface* of the three homothetic surfaces.

7. *Application.* Three concentric spheres whose radii are a, b, c are evidently homothetic to one another. We take the centre of the spheres for the origin of a rectangular system. Denoting by (θ, φ) the spherical coordinates of a point on the sphere a, we shall have

$$x = a \cos\theta \cos\varphi, \quad y = a \cos\theta \sin\varphi, \quad z = a \sin\theta,$$

and similarly for points on other spheres.

The coordinates of the point of intersection of the planes drawn through three corresponding points parallel to the axial planes are

$$x_1 = a \cos\theta \cos\varphi, \quad y_1 = b \cos\theta \sin\varphi, \quad z_1 = c \sin\theta,$$

whence we deduce the relation

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 + \left(\frac{x_1}{c}\right)^2 = 1.$$

Hence the locus of the point (x_1, y_1, z_1) is an ellipsoid with axes equal to $2a$, $2b$, $2c$.

The volume of this ellipsoid is therefore equal to the geometrical mean of

$$\frac{4}{3}\pi a^3, \quad \frac{4}{3}\pi b^3, \quad \frac{4}{3}\pi c^3,$$

that is, to $\frac{4}{3}\pi abc$.

G. SAWATA, ON THE ASYMPTOTIC LINES OF A SURFACE AND STRAIGHT LINES ON IT.[1)]

The equation of an asymptotic line will be constructed in the following manner:

The direction cosines λ, μ, ν of the inflection tangent (three-points-tangent) to the surface $\varphi(x, y, z) = 0$ at the point (x, y, z) satisfy the equations

$$(1) \qquad \lambda U + \mu V + \nu W = 0,$$

$$(2) \qquad \lambda^2 a + \mu^2 b + \nu^2 c + 2\mu\nu f + 2\nu\lambda g + 2\lambda\mu h = 0,$$

where U, V, W and a, b, c, f, g, h are the partial differential coefficients of the 1^0 and 2^0 orders of φ.

λ, μ, ν are also the direction cosines of the tangent to the asymptotic line, so that

$$(3) \qquad \lambda : \mu : \nu = dx : dy : dz,$$

where dx, dy, dz are the differentials at any point of the asymptotic line.

Hence the equations to the asymptotic line are got from (1), (2), (3), by a combination, in the form

$$(4) \qquad \frac{dx}{dz} = \frac{\lambda}{\nu} = \frac{UWb + V(-Uf + Vg - Wh) \pm V\sqrt{D}}{2UVh - U^2 b - V^2 a},$$

$$(5) \qquad \frac{dy}{dz} = \frac{\mu}{\nu} = \frac{VWa + U(Uf - Vg - Wh) \pm U\sqrt{D}}{2UVh - U^2 b - V^2 a},$$

where D represents the determinant

$$D = \begin{vmatrix} a & h & g & U \\ h & b & f & V \\ g & f & c & W \\ U & V & W & 0 \end{vmatrix}.$$

1) The Journal of the Mathematico-Physical Society in Tokyo, Vol. IV, pp. 186—212.

In June, 1888, the Mathematico-Physical Society in Tokyo proposed a prize essay that was to relate to a study on the relation between asymptotic lines and straight lines on a surface. The essay was to be sent to the committee consisting of D. Kikuchi, H. Terao, and R. Fujisawa on or before March 1, 1889.

There was a single paper sent to the committee, namely the one before us.

According to the report of the committee, the results arrived at by G. Sawata are by no means new discoveries, being already given by Clebsch, Salmon, and others. Besides the methods employed are much more complicated than followed

One of these equations is derivable from the other and $\varphi = 0$. Hence we take as the equations of the asymptotic line one of these together with $\varphi = 0$. The constant of integration is the parameter of the system of such lines.

For the ambiguous signs $\pm$ there correspond two such systems, which are real or imaginary according as $D >< 0$. Therefore the actual asymptotic lines find themselves only in the domain where the original surface intersects with $D = 0$, that is the locus of parabolic points. In such a domain the two systems of asymptotic lines come with each other along common tangents.

In the surface for which everywhere $D = 0$, that is, in developable surfaces, the asymptotic lines are all straight lines.

The equations (4) and (5) may be written in the symmetrical form

$$(6) \qquad \frac{dx}{P+N+M \pm (V-W)\sqrt{D}} = \frac{dy}{N+Q+L \pm (W-U)\sqrt{D}} = \frac{dz}{M+L+R \pm (U-V)\sqrt{D}},$$

where P, Q, R, L, M, N are the first minors of D in regard to a, b, c, f, g, h.

Straight lines that exist on a surface are necessarily contained among the asymptotic lines. Hence in order to find the straight lines we have to see for what values of the parameter the asymptotic lines become straight lines. If $x = f_1(z, p)$, $y = f_2(z, p)$, with p as a parameter, represent the asymptotic lines, then $f_3(p)$ being the highest common divisor of $\frac{\partial^2 f_1}{\partial z^2}$ and $\frac{\partial^2 f_2}{\partial z^2}$ independent of z, the roots of $f_3(p) = 0$ are the required values of the parameter. If there be no such common divisor, there will be no straight line on the surface.

But this way of reasoning is applicable only when the integral equations of the curve are found. We must look for another way. For such a purpose it is convenient to consider curves described by points where tangents can be drawn through four consecutive points and to study straight lines by their means. A tangent of this kind is called a *four-points tangent*.

by his predecessors. But in spite of the entangements of his manipulations, the author has well led his way and has got at his final goal. Thus G. Sawata was awarded with the prize for his pains on the general meeting of the Society on the 4th of May, 1889.

Thus we rewrite the equations (4) and (5) with the notations employed in (6),

$$\text{(7)} \qquad \frac{dx}{dz} = \frac{\lambda}{\nu} = \frac{M \pm V\sqrt{D}}{R} = I, \quad \text{say,}$$

$$\text{(8)} \qquad \frac{dy}{dz} = \frac{\mu}{\nu} = \frac{L \pm U\sqrt{D}}{R} = J, \quad \text{say.}$$

Since a point on an asymptotic line with a stationary tangent is a point on this surface where a four-points tangent can be drawn, we get for such a point

$$\text{(9)} \qquad I' = \lambda\frac{\partial I}{\partial x} + \mu\frac{\partial I}{\partial y} + \nu\frac{\partial I}{\partial z} = 0,$$

$$\text{(10)} \qquad J' = \lambda\frac{\partial J}{\partial x} + \mu\frac{\partial J}{\partial y} + \nu\frac{\partial J}{\partial z} = 0.$$

These two equations are not independent of one another, which can be proved in various ways. We proceed to show it analytically.

The values of P, Q, R, L, M, N as well as the determinant D being differentiated in respect to x, y and z, and also the derived functions of such expressions as $U^2D = L^2 - QR$, etc., which can be easily established, being formed, the following identities make establish themselves from these results:

$$\text{(11)} \qquad \begin{cases} UP_i + VN_i + WM_i = 0, \\ UN_i + VQ_i + WL_i = 0, \\ UM_i + VL_i + WR_i = 0, \end{cases}$$

$$\text{(12)} \quad P_1 + N_2 + M_3 = 0, \quad N_1 + Q_2 + L_3 = 0, \quad M_1 + L_2 + R_3 = 0,$$

$$\text{(13)} \qquad \begin{cases} U^2D_i = 2LL_i - QR_i - RQ_i, \\ V^2D_i = 2MM_i - RP_i - PQ_i, \\ UVD_i = RN_i + NR_i - LM_i - ML_i, \end{cases}$$

where i stand for 1, 2, 3, and where P_1, P_2, P_3 are the same as P except that, c, b, f contained there are replaced by the differential coefficients of these quantities taken in respect to x, y, z respectively, and the same for the remaining quantities.

We have from (7) $RI = M \pm V\sqrt{D}$, which being differentiated with respect to x, y, z respectively, the results will be substituted in I' in (9), or

$$\nu\left(\frac{M \pm V\sqrt{D}}{R}\frac{\partial I}{\partial x} + \frac{L \mp U\sqrt{D}}{R}\frac{\partial I}{\partial y} + \frac{\partial I}{\partial z}\right),$$

and we get, after transformation by means of (12), (13) and those formulae that serve for the establishment of these,

$$(14)\qquad I' = \frac{v}{R} V\Theta = \frac{V}{R\sqrt{\{(M \pm V\sqrt{D})^2 + (L \mp U\sqrt{D})^2 + R^2\}}} \cdot \Theta$$

where

$$\Theta = 3\,\frac{VD_1 - UD_2}{2} - 2\,\frac{VR_1 - UR_2}{R}\,D$$
$$\pm \sqrt{D}\left\{\frac{MD_1 + LD_2 + RD_3}{2D} - 2\,\frac{MR_1 + LR_2 + RR_3}{R}\right\}$$

with

$$D_1 = a_1 P + b_1 Q + c_1 R + 2\theta L + 2a_3 M + 2a_2 N, \quad \text{etc.},$$

the following designations being employed,

$$a_1 = \frac{\partial^3 \varphi}{\partial x^3}, \quad a_2 = \frac{\partial^3 \varphi}{\partial x^2 \partial y}, \quad a_3 = \frac{\partial^3 \varphi}{\partial x^2 \partial z},$$

$$b_1 = \frac{\partial^3 \varphi}{\partial x \,\partial y^2}, \ldots, c_1 = \frac{\partial^3 \varphi}{\partial x\, \partial z^2}, \ldots, \theta = \frac{\partial^3 \varphi}{\partial x\, \partial y\, \partial z}.$$

In like manner we have for J'

$$J' = \frac{U}{R\sqrt{\{(M \pm V\sqrt{D})^2 + (L \mp U\sqrt{D})^2 + R^2}} \cdot \Theta.$$

Hence for $\Theta = 0$, both I' and J' simultaneously vanish, and conversely I' and J' vanish together only when $\Theta = 0$.

For, if U and V might have a common factor, the denominator also contains U or V, so that I' and J' do not vanish in general, when U and V do. Therefore the intersection of the surface $\Theta = 0$ and the original surface is the locus of points where four-points tangents can be drawn. This equation has however a superfluous factor, which we next investigate.

By squaring we clear the equation $\Theta = 0$ of its radical sign, and after reduction we obtain

$$(15)\quad \frac{RT}{8D} - D_1(MM_1 + LM_2 + RM_1) - D_2(ML_1 + \cdots) - D_3(MR_1 + \cdots)$$
$$+ 2D(M_1^2 + L_2^2 + L_1 M_2 + M_1 L_2 + R_1 M_3 + R_2 L_3) = 0,$$

where

$$(16)\quad T = PD_1^2 + QD_2^2 + RD_1^2 + 2LD_2 D_3 + 2MD_3 D_1 + 2ND_1 D_2.$$

The quantities D_1, M_1, M_2, etc., being substituted by their values, the equation will be found to contain the factor R, which is evidently superfluous for our purpose. This factor being removed the remaining part will then assume the form

$$(17)\qquad -\frac{T}{8D} + E = 0,$$

where

$$\begin{aligned}E = P\{&U^2(\theta^2-b_1c_1)+V^2(a_3{}^2-c_1a_2)+W^2(a_2{}^2-a_1b_1)\\&+2VW(a_1\theta-a_3a_2)+2WU(b_1a_3-a_2\theta)+2UV(c_1a_2-\theta a_3)\}\\&+Q\{U^2(b_3{}^2-b_2c_2)+\cdots\}+\text{ etc.}\end{aligned}$$

In this expression the coefficients of V^2 may be written in the form of two determinants:

$$-\begin{vmatrix}0&\theta&b_3&c_2&0\\\theta&a&h&g&U\\b_3&h&b&f&V\\c_2&g&f&c&W\\0&U&V&W&0\end{vmatrix}+\begin{vmatrix}0&b_1&b_2&b_3&0\\c_1&&&&\\c_2&&&''&\\c_3&&&&\\0&&&&\end{vmatrix}.$$

The coefficients of $2VW$ and other terms are all expressible as aggregates of like determinants. Therefore the expression for E consists from twelve of these determinants. We have also for T

$$T=-\begin{vmatrix}0&D_x&D_y&D_z&0\\D_x&&&&\\D_y&&&&\\D_z&&''&&\\0&&&&\end{vmatrix}.$$

Thus the equation (17) may be written with a determinant on its one side and with twelve on the other, the terms being all multiplied with partial differential coefficients of φ. The equation arranged in such a form, after a transformation, results in the following:

$$(18)\qquad \frac{1}{4D}\begin{vmatrix}0&D_1&D_2&D_3&0\\D_1&&&&\\D_2&&''&&\\D_3&&&&\\0&&&&\end{vmatrix}=\begin{vmatrix}0&a_1&a_2&a_3&0\\P_1&&&&\\P_2&&''&&\\P_3&&&&\\0&&&&\end{vmatrix}+\cdots,$$

where the right-hand member consists of six determinants, whose first lines and first columns are made up, instead of a_1, a_2, a_3 and P_1, P_2, P_3, in the first term, of b_1, b_2, b_3; c_1, c_2, c_3; θ, b_3, c_2; a_3, θ, c_1; a_2, b_1, θ; and P_1, P_2, P_3; Q_1, Q_2, Q_3; R_1, R_2, R_3; L_1, L_2, L_3; M_1, M_2, M_3; N_1, N_2, N_3 respectively; and the last three determinants are multiplied by the coefficient 2. *This is the general formula for the curve of four-points tangents.*

In the following we confine our attention to the consideration of *an algebraical surface of degree k.* We here introduce a fourth variable ϱ for the purpose of rendering expressions symmetrical.

Designate $\frac{\partial \varphi}{\partial \varrho}$ by Π and the differential coefficients of this quantity in respect to x, y, z and ϱ by l, m, n, s, those of $\frac{\partial^2 \varphi}{\partial \varrho^2}$ by s_1, s_2, s_3, s_4 and those of $\frac{\partial \varphi}{\partial \varrho}$ of the second order with regard to x, y, z by $a_4, b_4, c_4, \theta_1, \theta_2, \theta_3$. We then have

$$\begin{aligned} Ux + Vy + Wz + \Pi\varrho &= k\varphi, \\ ax + hy + gz \;\; + l\varrho \;\; &= (k-1)\,U, \\ hx + by + fz \;\; + m\varrho &= (k-1)\,V, \ldots \end{aligned}$$

If Δ denote the Hessian of φ, it follows

$$(k-1)^2 D = \varrho^2 \Delta - k(k-1)s\varphi. \tag{19}$$

Transforming P, Q, R, etc., by the above formulae, we get

$$\begin{aligned} (k-1)^2 P &= -k(k-1)\xi\varphi + x^2 S - 2x\varrho X + \varrho^2 A, \\ (k-1)^2 Q &= -k(k-1)\eta\varphi + y^2 S - 2y\varrho\, Y + \varrho^2 B, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where the Greek letters stand for certain expressions and where A, B, C, etc., are the first minors of Δ.

These equations being differentiated, it follows from them

$$(k-1)^2 P_1 = (k-1)(k-2)\xi U - 2(k-2)(xS - \varrho X) + x^2 S_1 - 2x\varrho X_1 - \varrho^2 A_1,$$

and like formulae, where A_1, A_2, A_3, A_4, etc., are employed for the differential coefficients of A and the remaining quantities. Here the terms that contain φ have been postponed, because we shall come no more to take recourse to their differentiations and because the result solely lies on the intersection with the original surface.

Let p and q be any algebraical functions of the degrees i and j respectively, and let their partial differential coefficients be denoted by $p_1, p_2, p_3, p_4, q_1, \ldots$, then taking recourse to the relations

$$\begin{aligned} p_1 A + p_2 H + p_3 G + p_4 X &= p', \\ p_1 H + p_2 B + p_3 F + p_4 Y &= p'', \\ &\ldots\ldots\ldots\ldots \\ p_1 X + p_2 Y + p_3 Z + p_4 S &= \dot{p}, \end{aligned}$$

$$\begin{aligned} p_1 q' + p_2 q'' + p_3 q''' + p_4 \dot{q} &= q_1 p' + q_2 p'' + q_3 p''' + q_4 \dot{p} \\ &= \psi(pq) = \psi(qp), \end{aligned}$$

$$\begin{vmatrix} 0 & q_1 & q_2 & q_3 & 0 \\ p_1 & & & & \\ p_2 & & ,, & & \\ p_3 & & & & \\ 0 & & & & \end{vmatrix} = [pq],$$

we get

$$(20) \qquad (k-1)^2[pq] = \varrho^2\psi(pq) + ip\dot{q} + j\varrho q\dot{p} - ijpqS,$$

where φ has been put to 0 as before, and where $a', b', \ldots$ are connected with the relations

$$\begin{aligned} xa' + yb' + zg' + \varrho l' &= (k-2)\Delta, \\ xh' + yb' + zf' + \varrho m' &= 0, \\ \cdots\cdots&\cdots\cdots \end{aligned}$$

The first term in (18), that will be represented by $[Pa]$ according to our notation, can be transformed into

$$\begin{aligned} (k-1)^4[Pa] = &-2(k-2)(xS-\varrho X)\{-x\varrho a + \varrho^2 a' + (k-2)a(xS-\varrho X)\} \\ &+ (k-1)^2\{x^2[Sa] - 2x\varrho[Xa] + \varrho^2[Aa]\}, \end{aligned}$$

and similarly for $[Lf]$, $[Qb]$, etc.

But the right-hand member of the same formula may be written in the form

$$\begin{aligned} -2E = [Pa] + [Nb] + [Mg] + [Nh] \\ + [Qb] + [Lf] + [Mg] + [Lf] + [Rc]. \end{aligned}$$

This being multiplied with $(k-1)^4$ and the terms without square brackets in the above formulae being substituted, we obtain after transformation

$$\sigma(k-2)^2\varrho^2 S\Delta - 2(k-2)\varrho^3\dot{\Delta}.$$

Again the substitution of the terms within the square brackets being made in the same quantity, we transform and simplify the result, which we add to the above expression, when we arrive at

$$-2(k-1)^4 E = -4(k-2)^2\varrho^2 S\Delta + 2(k-2)\varrho^3\Delta - \varrho^4\Omega,$$

where

$$\Omega = \psi(Aa) + \psi(Bb) + \psi(Cc) + 2\psi(Ff) + 2\psi(Gg) + 2\psi(Hh).$$

From (19) by differentiation and transformation we get three formulae, from which follows

$$-(k-1)^4 T = \frac{\varrho^4}{(k-1)^2}\{-\varrho^2\psi(\Delta\Delta) + 8(k-2)\varrho\Delta\dot{\Delta} - 16(k-2)^2\Delta^2 S\}.$$

Thus the equation $\frac{T}{4D} - 2E = 0$ transforms into

$$\psi(\Delta\Delta) - 4\Delta\Omega = 0, \tag{21}$$

which is *the equation, reduced to its simplest form, of the curve of the four-points tangents.*

The equation to the curve of four-points tangents is of the degree $(11k - 24)$. For Δ is of the degree $4(k-2)$, and $a_1, a_2, \ldots$ are all of the degree $k-3$, so that $\psi(\Delta\Delta)$ and $\Delta\Omega$ are each of the degree $11k - 24$. And a test for a certain special term will show that its degree cannot be reduced any more.

As Clebsch and others have said, in a surface of the third degree, the curve of four-points tangents is of course a straight line. In this case, $k = 3$, so that the equation (21) is of the 9th degree. From this equation with that of the original surface we see that the surface contains 27 straight lines on it.

As to the straight lines on a surface of the 4th and higher degrees, if any, they will be contained among the curves of the four-points tangents. Hence *the number of straight lines on a surface of degree k does not surpass* $(11k-24)k$. *If surpass, then an infinite number of them.*

The equation to the curve of four-points tangents serves to discover the straight lines. If $f(x, y, z)$ denote the curvature of the curve, when the surfaces $\varphi = 0$, $f = 0$ and that represented by the equation to the curve intersect along a common curve, this is a straight line. This fact may be tested in various ways, as one of which we may see whether the solution of the three equations is indeterminate or not. As another one we may project the two curves got from the three equations on some one plane and see they have common factor or not.

In practice it will be more convenient not to find the curvature of the curve, but to project it directly on a plane, the xy plane for instance. If $f_1(x, y) = 0$ is this projection and $f_2(x, y)$ its curvature, we have to find the highest common factor $f_3(x, y)$. Then $f_3(x, y) = 0$ is necessarily a straight line. But this fact cannot be always ascribed to the existence of a straight line on the surface, for the projection of a curve might sometimes happen to be a straight line. Hence the intersection of $f_3(x, y) = 0$ and $\varphi = 0$ should be projected on another plane, xz plane say, resulting in the curve $f_4(x, z) = 0$, whose curvature

is $f_5(x, z)$. The highest common factor of $f_4(x, z)$ and $f_5(x, z)$ should be found in the form $f_6(x, z)$. Then the intersection of $f_6(x, z) = 0$ and $\varphi = 0$ is a straight line that exists on the surface. If there be no such common factor, it is evident that there is no straight line on the surface.

T. FUJII, ON THE PERIMETER OF AN ELLIPSE.[1])

An ellipse may be conceived as the projection of a circle upon a plane. Take a circle whose radius is equal to a and project it orthogonally on a plane, that passes through a diameter AB of the circle. The projection is an ellipse with the semi-axes a and b. Let the inclination of the two planes be denoted by β. Then $\sin^2\beta = \frac{a^2 - b^2}{a^2}$.

On the circular quadrant AA_n, take $n - 1$ points $A_1, A_2, A_3, \ldots, A_{n-1}$, and from these points draw A_1a_1, A_2a_2, ... at right angles to AB.

The distances from the points A_1, A_2, ... to the plane of the ellipse are

$$A_1a_1 \sin\beta = A_1a_1\sqrt{\frac{a^2-b^2}{a^2}}, \quad A_2a_2\sqrt{\frac{a^2-b^2}{a^2}}, \ldots$$

Hence the projections of AA_1, A_1A_2, A_2A_3, ... may be easily calculated, and their sum is equal to

$$\sqrt{AA_1^2 - A_1a_1^2 \cdot \frac{a^2-b^2}{a^2}} + \sqrt{A_1A_2^2 - (A_2a_2 - A_1a_1)^2\frac{a^2-b^2}{a^2}}$$
$$+ \cdots + \sqrt{A_{n-1}A_n^2 - (A_na_n - A_{n-1}a_{n-1})^2\frac{a^2-b^2}{a^2}},$$

or expanded by the binomial theorem,

$$= (AA_1 + A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n)$$
$$-\frac{a^2-b^2}{a^2}\left\{\frac{A_1a_1^{\,2}}{AA_1} + \frac{(A_2a_2 - A_1a_1)^2}{A_1A_2} + \cdots + \frac{(A_na_n - A_{n-1}a_{n-1})^2}{A_{n-1}A_n}\right\}$$
$$-\frac{3(a^2-b^2)^2}{2\cdot 4a^4}\left\{\frac{A_1a_1^{\,4}}{AA_1^{\,3}} + \frac{(A_2a_2 - A_1a_1)^4}{A_1A_2^{\,3}} + \cdots + \frac{(A_na_n - A_{n-1}a_{n-1})^4}{A_{n-1}A_n^3}\right\} - \cdots$$

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1) The Journal of the Society of Mathematics in Tokyo, Vol. 4, pp. 550—553, 1891.

For simplicity's sake we put

$$AA_1 = A_1A_2 = A_2A_3 = \cdots = A_{n-1}A_n = A,$$

$$A_1a_1 = a_1,\ A_2a_2 = a_2,\ \ldots,\ A_{n-1}a_{n-1} = a_{n-1},\ \text{and}\ \frac{a^2-b^2}{a_2} = \alpha^2,$$

and let 2θ denote the angle contained by the successive circular chords. Then evidently

$$a_1 = A\sin\theta,\ \ a_2 - a_1 = A\sin 3\theta,\ \ \ldots,\ \ a_n - a_{n-1} = A\sin(2n-1)\theta,$$

and the preceding expression will become

$$A\Big[n - \frac{\alpha^2}{2}\Big\{\sin^2\theta + \sin^2 3\theta + \sin^2 5\theta + \cdots + \sin^2(2n-1)\theta\Big\}$$
$$- \frac{3\alpha^4}{2\cdot 4}\Big\{\sin^4\theta + \sin^4 3\theta + \cdots\Big\} - \frac{3\cdot 5\alpha^6}{2\cdot 4\cdot 6}\Big\{\sin^6\theta + \cdots\Big\} - \cdots$$

When n increases without limit, the totality of the inscribed chords tends to approach to the circular quadrant, so that their projections approach to the elliptic quadrant.

We can therefore, by calculating the values of some number of first terms in the last expression for a sufficiently large value of n, deduce the value of the elliptic perimeter as correctly as we please.

N. YAMAMOTO, ON A PROBLEM IN THE THEORY OF CONIC SECTIONS.[1]

For the points, real or imaginary, P_1, P_2, P_3, ..., P_n, where a curve of the n^{th} degree, $f(x, y) = 0$, is met by a straight line, $y = \alpha x + \beta$, we shall have

$$f(x, \alpha x + \beta) = A(x - p_1)(x - p_2)\ldots(x - p_n) = 0,$$

which is of the degree n in x.

If $A(a, a')$ and $B(b, b')$ be two points on the straight line, then the coordinates of these points will satisfy the identities

$$f(a, a') = A(a - p_1)(a - p_2)\ldots(a - p_n),$$
$$f(b, b') = A(b - p_1)(b - p_2)\ldots(b - p_n),$$

whence

$$\frac{f(a, a')}{f(b, b')} = \frac{a - p_1}{b - p_1}\cdot\frac{a - p_2}{b - p_2}\cdot\ \ldots\ \cdot\frac{a - p_n}{b - p_n}.$$

1) The Journal of the Society of Mathematics, Vol. 6, pp. 253—256, 1893.

But evidently

$$\frac{a-p_1}{b-p_1}=\frac{AP_1}{BP_1},\quad \frac{a-p_2}{b-p_2}=\frac{AP_2}{BP_2},\ \ldots$$

$$\therefore\quad \frac{f(a,a')}{f(b,b')}=\frac{AP_1\cdot AP_2\cdot\ \cdots\ \cdot AP_n}{BP_1\cdot BP_2\cdot\ \cdots\ \cdot BP_n},\quad\text{or}\quad =\frac{[AP]}{[BP]}.$$

Now we consider three transversals which form a triangle ABC.

Let the points of intersection of the curve $f=0$ with the lines AB, BC and CA be $P_1, P_2, \ldots, P_n$; $Q_1, Q_2, \ldots, Q_n$; $R_1, R_2, \ldots, R_n$ respectively. Then the coordinates of A, B, C being (a, a'), (b, b'), (c, c'), we have from the above formula

$$\frac{f(a,a')}{f(b,b')}=\frac{[AP]}{[BP]},\quad \frac{f(b,b')}{f(c,c')}=\frac{[BQ]}{[CQ]},\quad \frac{f(c,c')}{(f\,a,a')}=\frac{[CR]}{[AR]},$$

whence by multiplication we get

$$[AP]\,[BQ]\,[CR]=[BP]\,[CQ]\,[AR].$$

In the case of a conic this formula will assume the form

$$AP\cdot AP'\cdot BQ\cdot BQ'\cdot CR\cdot CR'=BP\cdot BP'\cdot CQ\cdot CQ'\cdot AR\cdot AR'.$$

When the points P, Q, R become coincident with P', Q', R' respectively, the three transversals will each become a tangent to the conic, and the last formula takes the form

$$AP^2\cdot BQ^2\cdot CR^2=BP^2\cdot CQ^2\cdot AR^2,$$

or

$$AP\cdot BQ\cdot CR=BP\cdot CQ\cdot AR.$$

Thus the three lines AQ, BR, CP are concurrent.

We infer therefore that *in a triangle whose three sides touch a conic, the three straight lines, that join the vertices with their opposite points of contact, are concurrent.*

G. SAWATA,
HOW TO DRAW HIGHER ALGEBRAIC CURVES.

On this subject there are two papers published by G. Sawata in *the Journal of the Mathematico-Physical Society in Tokyo*, Vol. VI, pp. 11—18 and Vol. VII, pp. 13—17, January and August, 1895. The first concerns the description of the curves of the fourth degree, while the latter treats of drawing curves whose degrees are higher than the fourth.

I.

On the description of curves of the fourth degree.

The analytical way of drawing the figures of curves of the fourth degree can be usually effected only through intricated calculations, and the problem is by no means an easy matter. But here we propose to give a simple method of geometrical description and construction of such a curve, by which we enjoy the convenience of roughly seeing what a form the curve would have.

§ 1. *The transformation of the equation.* The general equation of the curve of the fourth degree

$$(1) \qquad \begin{aligned} c_1 x^4 + c_2 x^3 y + c_3 x^2 y^2 + c_4 x y^3 + c_5 y^4 \\ + c_6 x^3 + c_7 x^2 y + c_8 x y^2 + c_9 y^3 \\ + c_{10} x^2 + c_{11} x y + c_{12} y^2 \\ + c_{13} x + c_{14} y \\ + c_{15} = 0 \end{aligned}$$

can be transformed into the form of

$$(2) \qquad \varphi_1(xy)\,\varphi_2(xy) = \psi(xy),$$

where

$$\varphi_1(xy) = A_1 x^2 + 2H_1 xy + B_1 y^2 + 2G_1 x + 2F_1 y + C_1,$$
$$\varphi_2(xy) = A_2 x^2 + 2H_2 xy + B_2 y^2 + 2G_2 x + 2F_2 y + C_2,$$
$$\psi(xy) = K(x^2 + y^2) + 2Px + 2Qy + R,$$

with A_1, A_2, K, ... as constants. The number of the new constants is one more than the old; but φ_1 and φ_2 entering only in the form of a product, this transformation is unique. To obtain the transformed equation we first factorise the part in the highest degree, and get at

once the coefficients $A_1, A_2, H_1, H_2, B_1, B_2$. The remaining coefficients will be found by actually constructing the equation in the form of (2) and comparing it with (1).

In what follows we will call $\varphi_1 = 0$ and $\varphi_2 = 0$ the *basal conics*, and $\psi = 0$ the *basal circle.*

§ 2. *The values of the quadratic expressions.* The locus of a point, where the quadratic expression $\varphi(xy)$ has a value equal to a certain number c, has $\varphi(xy) = c$ for its equation. The two conics $\varphi(xy) = c$ and $\varphi(xy) = 0$ are evidently concentric and similar and similarly situated. Now assuming $\varphi = 0$ to be an ellipse, we denote by a and a' the like semi-axes (for instance the major semi-axes) of these conics, and we have

$$a^2 = -k\varphi(x_0y_0), \quad a'^2 = -k\{\varphi(x_0y_0) - c\},$$

where (x_0y_0) is the common centre, and where

$$k = \frac{1}{AB - H^2}\left\{\frac{A+B}{2} \pm \sqrt{\left(\frac{A-B}{2}\right)^2 + H^2}\right\}.$$

Hence we have

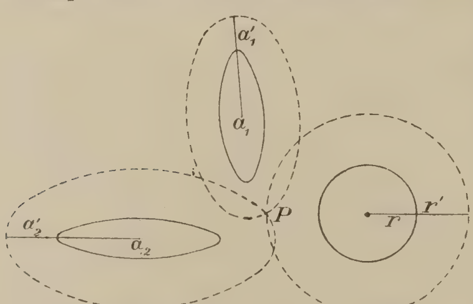
$$\text{(3)} \qquad \varphi(xy) = c = \frac{1}{k}(a'^2 - a^2).$$

Thus the value of $\varphi(xy)$ is expressed in terms of the semi-axes of the similar conics.

§ 3. *The constitution of curves of the fourth degree explained.* From § 2 we have

$$\varphi_1 = \frac{1}{k_1}(a_1'^2 - a_1^2), \quad \varphi_2 = \frac{1}{k_2}(a_2'^2 - a_2^2), \quad \psi = k(r'^2 - r^2),$$

where a_1 and b_2 represent like semi-axes of the basal conics and a_1' and a_2' those of the conics similar to them, and r, r' the radii of the basal circle and one concentric with it. Hence from the equation of our curve we deduce



$$\text{(4)} \quad (a_1'^2 - a_1^2)(a_2'^2 - a_2^2) = k'(r'_2 - r^2),$$

where $k' = k_1k_2k = \text{const.}$

(A) *The curve, therefore, may be considered as the locus of a point for which $(a_1'^2 - a_1^2)(a_2'^2 - a_2^2)$ is proportional to $(r'^2 - r^2)$, when the curves concentric and similar and similarly situated with the three basal curves are drawn passing through a common point.*

(B) Again *the curve may be treated as the locus of a point for which* $\left\{\left(\frac{r_1'}{r_1}\right)^2 - 1\right\}\left\{\left(\frac{r_2'}{r_2}\right)^2 - 1\right\}$ *is proportional to* $\left\{\left(\frac{r'}{r}\right)^2 - 1\right\}$, *where* r_1', r_2', r *are the lengths of straight lines drawn from a point to the centres of the basal curves and where* r_1, r_2, r *are the distances of their centres to their intersections with these lines.*

For from (4), divided by $a_1^2 a_2^2$, we have

$$\left\{\left(\frac{a_1'}{a_1}\right)^2 - 1\right\}\left\{\left(\frac{a_2'}{a_2}\right)^2 - 1\right\} = \frac{k' r^2}{a_1^2 a_2^2}\left\{\left(\frac{r'}{r}\right)^2 - 1\right\},$$

and since $\frac{a_1'}{a_1} = \frac{r_1'}{r_1}$, $\frac{a_2'}{a_2} = \frac{r_2'}{r_2}$, we also have

$$\left\{\left(\frac{r_1'}{r_1}\right)^2 - 1\right\}\left\{\left(\frac{r_2'}{r_2}\right)^2 - 1\right\} = k''\left\{\left(\frac{r'}{r}\right)^2 - 1\right\}, \tag{5}$$

where $k'' = \frac{k' r^2}{a_1^2 a_2^2} = \text{const.}$

§ 4. *The construction.* Now we can utilise what we have said in (A) in § 3 in actually carrying the construction of our curve.

Two series of conics concentric with the two basal conics and similar and similarly situated to them should be described in such a way that in each series the differences of the squares of like semi-axes of the similar conics and the basal conic are in a geometric progression, whose common ratio is the same for both series. The intersections of curves in these two series form groups of points where $\varphi_1\varphi_2$ *has equal values. The points of each group being joined in order, we obtain a series of curves whose equation is* $\varphi_1\varphi_2 = c$, *the values of* c *proceeding in a series with the same common ratio as above mentioned. The series of concentric circles* $\psi = c$ *should also be described. The intersections of these circles and the corresponding members of the series* $\varphi_1\varphi_2 = c$ *being joined in order give the form of our curve.*

When we are required to draw a part of our curve in a high degree of accuracy, we can advantageously employ (B) in last article. At a point that is suspected to lie on the curve, the quantities

$$\left\{\left(\frac{r_1'}{r_1}\right)^2 - 1\right\}\left\{\left(\frac{r_2'}{r_2}\right)^2 - 1\right\} \text{ and } k''\left\{\left(\frac{r'}{r}\right)^2 - 1\right\}$$

being calculated and compared, we can determine whether that point is, or is not, on the curve. In case when it does not belong to it, it can be determined on which side of the curve it lies.

The above way of construction being merely graphical, it cannot be conducted very minutely in the vicinities of double points, so that for such portions of the curve the usual analytical achievement will be recommended.

II.

On drawing curves of higher degrees.

The way we have given for the construction of quartic curves may be applied to the case of higher curves in general.

Curves of the fifth and higher degrees cannot be described but with a considerable amount of labour and with much hardships. For instance, to find the position of a point directly, we have to make repeated substitutions of severally assumed values in the series of Sturm's functions and thus examine the changes of the sign and finally arrive at a rough value, when we can go to apply for an approximate solution such as Horner's method would answer. The laborious efforts indispensable in such a calculation may well be imagined, and sometimes many an hour will be required for the determination of a single point. The way we propose here to describe is noway simple, but it may be sometimes found convenient, we don't doubt. It will be especially advantageous in those cases where merely rough figures are sought for.

§ 5. *The equation of a curve resolved.* An expression of the m^{th} degree in x and y can be represented as the sum of the continued product of $\frac{m}{2}$ quadratic expressions with an expression of the $(m-2)^{\text{th}}$ degree, when m is even, and as the sum of the continued product of a linear and $\frac{m-1}{2}$ quadratic factors with an expression of the $(m-2)^{\text{th}}$ degree, when m_1 is odd. In other words, let $F_r(xy)$ be an expression of the degree r, and let be denoted

$$\varphi_i = a_i x^2 + h_i xy + b_i y^2 + g_i x + f_i y + c_i,$$

and by $\Pi_r \varphi$ the continued product of $\varphi_1, \varphi_2, \ldots, \varphi_r$. Then we have, in the case $m = 2n$,

$$F_{2n}(xy) = \Pi_n \varphi + F_{2n-2}(xy), \tag{6}$$

and in the case $m = 2n + 1$,

$$F_{2n+1}(xy) = (Ax + By + C)\,\Pi_n \varphi + F_{2n-1}(xy). \tag{7}$$

We will first treat of *the case m even.*

The coefficients $a_1, h_1, b_1, \ldots$ in the factors of $\Pi_n \varphi$ are obtained by resolving the part of the highest degree in the expression into quadratic factors. Then g_1, f_1, etc., will be given by $2n$ linear equations which arise from the comparison of the part of the degree $2n-1$ in the expansion of $\Pi_n \varphi$ with that in the original expression, $c_1, c_2, \ldots$ are quite arbitrary.

Next decomposing $F_{2n-2}(xy)$ into the sum of $\Pi_{n-1}\varphi$ and $F_{2n-3}(xy)$, and proceeding in like manner, we arrive at

$$(8)\qquad F_{2n}(xy) = \Pi_n\varphi + \Pi_{n-1}\varphi + \cdots + \Pi_2\varphi + \psi,$$

where ψ is of the second degree. If the values of c_1 and c_2 in $\Pi_2\varphi$ are properly chosen, we can make

$$\psi = k(x^2 + y^2) + px + qy + R,$$

so that $\psi = 0$ should represent a circle.

§ 6. *The locus of $\Pi\varphi = c$.*

As we have described in Part I., at points on a conic concentric with and similar and similarly situated to $\varphi = 0$, the function φ has a constant value proportional to the difference of the squares of the like semi-axes of these conics. Hence it is very easy to find the semi-axis of any of the similar conics.

By describing series of similar conics for which the values of φ_1 and φ_2 are each in a geometric progression, and by joining their corresponding intersections, we get the series of the curve $\varphi_1\varphi_2 = c$. The values of $\varphi_1\varphi_2$ (that is, c) will also proceed in a G. P.

Similarly the series of $\varphi_3 = c$ (this c also in a G. P.) being described, and compounded with the series $\varphi_1\varphi_2 = c$, we shall obtain the series of $\varphi_1\varphi_2\varphi_3 = c$. Proceeding in like manner we arrive at the series of $\Pi_r\varphi = c$.

Thus we get the graphs of

$$\Pi_n\varphi = c, \quad \Pi_{n-1}\varphi = c, \ \ldots, \quad \Pi_2\varphi = c, \quad \psi = c.$$

§ 7. *The graphs of various continued products compounded.* By compounding the curves $\Pi_2\varphi = c$ and $\psi = c$ we obtain the series of the graphs of $\Pi_2\varphi + \psi = c$. The result being again compounded with graphs of $\Pi_3\varphi = c$, the series of $\Pi_3\varphi + \Pi_2\varphi + \psi = c$ is obtained. And in such a manner we get at last the curve

$$\Pi_n\varphi + \Pi_{n-1}\varphi + \cdots + \Pi_2\varphi + \psi = 0,$$

which is the curve required.

We have chosen the values of $\varphi_1, \varphi_2, \ldots$ to lie in G. P., because we require thereby to win the convenience for the description of the graphs of the products $\Pi\varphi$. But we acquire thereby also the convenience to have the successive curves a good way apart from one another and not coming too closely together.

In compounding the graphs of different continued products we are however little afforded convenience from the series distributed in G. P.

Accordingly in the process of compounding such graphs, it will be advantageous, if we conveniently describe other curves between the original curves of one series so as to form an A. P. For in the series thus changed in A. P., if we take a certain one among them as the first term, then there will be those that succeed as the following members of a G. P. Thus we can get the resulting curves proceeding in a G. P.

The last process of compounding has for its object to get a single curve, so that the two component series should be taken as they originally stand.

In the case when *the equation of the curve is of an odd degree*, $2n+1$, we decompose it in the form

$$(9)\quad F_{2n+1}(xy) = (A_1x + B_1y + C_1)\,\Pi_n\varphi + (A_2x + B_2y + C_2)\,\Pi_{n-1}\varphi + \cdots$$

The process we have to take for this case little varies from the former.

P. S. When the curve proposed for tracing is only of the fifth or sixth degrees, its construction will not be so complicated, but the higher the degrees by far the more complicated the constructions. But in practice we can form rough graphs of the continued products and compound them so as to get a general view of the required curve, and we can then minutely describe those parts we are required. Thus we can save much labour. In some curves we can also advantageously take a proper measure that will best adapt to the case. For instance the curve

$$(ax^2 + hxy + by^2 + gx + fy + c)^n = (Ax + By + C)^m$$

will be traced by compounding the two series of the similar conics, that represent the left-hand side, and of the straight lines, that represent the right-hand member.

A CONTROVERSY ON THE POLAR EQUATION.

I. *M. Endō, Ordinary errors on the polar equations of conics. The Journal of the Physics School in Tokyo, Vol. 8, June, 1899.*

As to the linear coordinates of a point and the equation of a curve referred to a linear system, the reasoning is usually carried very minutely and exactly, while the definitions and conventions for the polar coordinates appear in their basis to lack a sound and rigorous foundation. Consequently there reigns an erroneous view over the

equations of the conics, which are falsely expressed. But so far as we know such an error is never pointed out, and so we have to consider about that point.

We will first give a rigorous definition and convention for the polar system of coordinates and then make a study on their relations to the rectilinear system and set clear the inadequate points of the equations hitherto employed; we will lastly see how such a looseness of consideration has arisen and we will give correct expressions to the equations of the conics.

1. We draw from a fixed point a straight line in a fixed direction and call the point the *pole* and the line the *initial line.*

The position of a point P can be determined in respect to the initial line OX. For draw the straight line that goes through P and O. Then if the length of OP and the angle contained by OP and OX are given, the position of P will be entirely determinate. The letters r and θ denoting these quantities respectively, we term them the *polar coordinates* of the point. (Here r and θ will underlie to the rule of signs as usual.)

To a pair of given values of r and θ there will correspond a single point; but the coordinates of a point are by no means restrited to a single pair, for $(r,\ 2n\pi+\theta)$ and $\{-r,\ (2n+1)\pi+\theta\}$ express all the same point as (r, θ).

If there is no treatise that has ever given a perfect definition and convention for the polar coordinates, yet there is none at the same time that contradicts with what we have just said.

2. The equations $x = r\cos\theta$, $y = r\sin\theta$ have no reference to special positions of the point, and they hold if we change (r, θ) into $(r,\ 2n\pi+\theta)$ or $\{-r,\ (2n+1)\pi+\theta\}$, so that these relations are true in general.

3. The equation of the circle that passes through the pole and the point (ϱ, α) and with radius a is usually given in the form of $r - 2a\cos(\theta-\alpha) = 0$. But this is *not* the equation to the circle. For draw a straight line through the pole making the angle θ' with the initial line. This line intersects with the circle in two points, different or coincident. Hence we must have two values of r for any value of θ'. The equation (1) gives however only one value of r.

As the pole is a point on the circle, one might well argue, the one value of r will do to determine another point of intersection. But this is no mathematical way of argument. In the Cartesian coordinates the equation to a line that passes through a point necessarily

contains this point. Thus one equation, that will be satisfied in every case, can be referred to as the equation of the line, but nothing else.

The equation $r = a$ is usually taken for the equation of a circle with the pole as centre. But this is no equation of the circle. For the coordinates $(r, 2n\pi + \theta)$ of a point P on the circle, will satisfy it, but the coordinates $\{-r, (2n+1)\pi + \theta\}$ of the same point does not satisfy it.

The equation of a conic with a focus at the pole and an axis along the initial line is usually given in the form $\frac{l}{r} = 1 + e\cos\theta$. But this is not an equation to the conic. For the coordinates of a point P on the conic will be $(r, 2n\pi + \theta)$ or $\{-r, (2n+1)\pi + \theta\}$. If the former satisfy the equation, the latter does not satisfy it.

How such false results as the above could have been introduced here? The error comes in because the definition and convention for the polar coordinates are not completely established, and merely superficial considerations are followed. We have nothing to do with the false simplicity of results.

4. The polar equation of a curve may be correctly constructed by an independent way if we pay some appropriate attentions on the nature of the problem, but the most simple way will lie in deducing it from the equation in a rectangular system.

From the equation $x^2 + y^2 - 2cx - 2dy = 0$, that represents a circle passing the origin of a rectangular system, we deduce

$$r^2 - 2ar\cos(\theta - x) = 0,$$

the polar equation of the circle, which passes through the pole.

The equation $x^2 + y^2 = r^2$ corresponds to the polar equation $r^2 = a^2$.

From the equation $x^2 + y^2 = (l + ex)^2$, the focus at the origin being taken for pole we deduce the polar equation to the conic $r^2 = (l + er\cos\theta)^2$.

From the above we see that the equation to a circle is

$$r^2 - 2ar\cos(\theta - \alpha) = 0,$$

and not

$$r - 2a\cos(\theta - \alpha) = 0;$$

and that $r = a$ and $r = l + er\cos\theta$ are not equations to a circle and a conic, but these should be represented by

$$r^2 = a^2, \quad \text{and} \quad r^2 = (l + er\cos\theta)^2.$$

II. *T. Hayashi in the same Journal, July, 1899. On polar coordinates.*

The analytical representation of a geometrical figure has to fulfil *one-to-one correspondences;* the employment of the positive and negative signs only serves to make clear such a correspondence. This will be seen in the Cartesian geometry. In the polar geometry this one-to-one correspondence between the position of a point and its coordinates can be won if we do not necessarily distinguish between the different signs of the coordinates as in the Cartesian geometry. From the necessity of employing the positive and negative signs in the Cartesian geometry we cannot conclude the same for the polar geometry, in which we can safely take all magnitudes as always positive without thereby restraining any sense of its convention. Such will be perhaps most adequate for that system which presents a far different aspect than the other. To take the equation of a conic in the form $r^2 = (l + er\cos\theta)^2$ is no regitimate representation, it is moreover a double representation of one and the same curve. Such an usage must be avoided.

III. *O. Sudō's remark in the same Journal, July, 1899 (pp. 230—232).*

1. M. Endō maintains that the coordinates of a point may be represented in the two ways $(r, 2n\pi + \theta)$ and $\{-r, (2n+1)\pi + \theta\}$, and that his convention does not contradict with anything given in various treatises. But as we know some five or six works put r *as always taking the positive sign.*

2. Would it be necessary that there should be an interrelation between the convention for the signs of x and y in the Cartesian geometry and that for those of r and θ in the polar geometry? Nothing of the sort as we see; what necessity of distinguishing the opposite directions by different signs whereas we already employ θ for the same purpose? But it is a convention; if there be no necessity, yet nothing prevents its employment, when we can find some conveniences in it. We do not therefore necessarily oppose against M. Endō's convention.

3. Proceeding with the same reasoning, as M. Endō does, starting from his own convention, we must conclude that an equation $f(r, \theta) = 0$ does not represent a curve unless it still holds when we replace r and θ by $-r$ and $\theta + \pi$. If so, where does the benefit of the convention lie upon? What will then be represented by such an equation that does not stand for a curve? Cannot the equation represent any-

thing geometrical? How will have M. Endō to represent a transcendental curve in the polar coordinates?

Assume the equation $f(r, \theta) = 0$ to have been obtained in some way by considering a curve with respect to the coordinates of one of its points (r', θ') and let it be such that the values of r and θ deduced by continuous variations from r' and θ' all satisfy it, but no other values satisfy it. Would it be incorrect to call this the equation of the curve? When we obtain another equation $f_1(r, \theta) = 0$, that will be satisfied by $(-r', \pi + \theta')$, would it not be convenient to call this too the equation to the same curve, and thus to take the two equally to represent it, as if the coordinates of the same point are represented by (r, θ) as well as by $(-r, \pi + \theta)$? Would it not be convenient, for example, that the same circle should be represented by $r = a$ as well as by $r = -a$, and the same parabola both by $\frac{l}{r} = 1 - \cos\theta$ and $\frac{l}{r} = -(1 + \cos\theta)$? Of course the degrees of such equations thus obtained may be lower than the degrees of the curves.

4. M. Endō says that $r - 2a\cos(\theta - \alpha) = 0$ is not the equation to a circle. This is true, indeed, for the point on the circle that coincides with the pole does not satisfy it. But every other point on the circle well satisfies it, so that we can as well term this as the equation of the circle and use it for the study of the points on the circle other than the pole. This is a natural way of procedure and we find nothing inadequate in so doing. We often meet with examples of the same nature in the equations to higher curves when their multiple points are taken for the pole.

IV. *F. Sembon's opinion. (The same Journal, pp. 256—258, August, 1899.)*

It would be very proper to employ the positive and negative signs in the polar coordinates as M. Endō does. But as to his reasonings and to the equations of conics employed by him we maintain a different opinion.

The relations $x = r\cos\theta$, $y = r\sin\theta$ will be satisfied by the values r, $2n\pi + \theta$ as well as $-r$, $(2n+1)\pi + \theta$. Hence the polar equation deduced from a Cartesian equation will be satisfied by both pairs of values. But we can hardly say that an equation cannot represent a curve unless satisfied by both pairs of these values.

If r be admitted to possess a positive as well as a negative sign, the equation $\frac{l}{r} = 1 + e\cos\theta$ will generate the whole part of the conic

when θ changes from 0 to π or 2π, so that this will be certainly proper to be called the equation to the conic.

The equation $r^2 = (l + er\cos\theta)^2$ can be separated into two equations and each of them describes the whole conic when r may have the positive and negative signs. One of these can therefore be looked upon as representing the conic.

It is not only unnecessary to take the equations $r^2 = (l + er\cos\theta)^2$ and $r^2 = a^2$, but it is loathsome because we thus entangle ourselves in needless complications.

In case we don't allow the negative sign of r, the equation $\frac{l}{r} = 1 + e\cos\theta$ will, for $e > 1$, describe one branch and not the other branch of a hyperbola. If we have to represent the whole curve we must take another equation $\frac{l}{r} = -(1 - e\cos\theta)$. Hence if a single equation is to stand for the entire conic, or if all conics are to be represented by equations of the same form, we shall then be necessitated to write the equation in the form $r^2 = (l + er\cos\theta)^2$.

Negative values of r are necessary to be allowed even for a curve other than the conic if a single equation should suffice for its complete representation, and there happen also cases where positive and negative values of θ are to be referred to. There are of course many curves that can be represented by a single equation when θ has the positive values less than 2π and r only positive values; but such are special exceptions.

In this respect we agree with M. Endō's convention. The signs are employed in the Cartesian coordinates, for the entire curves are to be represented by single equations.

Still a few words. The equation $r - 2a\cos(\theta - \alpha) = 0$ can or cannot be considered as representing a circle according to the ways we have explained. In the polar system, the curve passes through the pole if the value of r vanish for some value of θ. The equation $r - 2a\cos(\theta - \alpha) = 0$ can therefore be considered as to represent the circle going through the pole; for the value of r vanishes for $\theta = \frac{\pi}{2} + \alpha$.

V. *M. Endō's reply. The same Journal, pp. 258—259, August, 1899.*

Our treatment on the relation between the polar and rectilinear coordinates appears to have been completely misunderstood by various mathematicians, notwithstanding we have purposely expressed to base our consideration on the sake of convenience. The relation between the two systems is nothing but a natural consequence of the definition

and convention, it is no result of artificial intermingling. We cannot agree to them on the point that the two systems are entirely independent of one another. Since both have the same subject for their objects, the results arrived at by them must not contradict each other.

The signs of the polar coordinates are no imitation of the rectilinear system; they are employed because they are required for the analytical representation of geometrical figures. If the positive and negative signs are not required, how it would be explained when a negative value of r corresponds to some value of θ? These values would be perhaps abandoned as meaningless, but we can see no cause in so doing. One would be too narrow minded when he has to put restraints on what could be easily got by making proper correspondences between the figure and its equation by means of a definition and convention.

As we see, the equation $f(r, \theta) = 0$ does not represent a curve on which any restriction is set, when it is not satisfied by replacing r and θ by $-r$ and $\pi + \theta$, but it represents a curve constructed according to some way. If we are required of the equation of a curve whose way of construction is not known, the equation must be satisfied by $-r$, $\pi + \theta$ as well as by r, θ. Such an equation is by no means in a double representation but in a perfect representation, and consequently not to be avoided. But we don't deny to consider the equation of a curve constructed in some way as the equation of the curve, for a convenience may be won in so doing. As to the question how the equation of transcendental curves should be constructed we have little wit to discern its meaning.

The equation $r - 2a \cos(\theta - \alpha) = 0$ may serve for the equation of a circle, if we conveniently tell of the matter, but it must not be done without setting any restraint. The same should be said about many other like cases.

M. KABA, A PROOF OF PASCAL'S THEOREM ON THE HEXAGRAM.[1]

If any six points be taken on a conic and each successive two of these points be joined so as to form a hexagram, then the intersections of the three pairs of opposite sides will be collinear.

This is a theorem usually known by the name of Pascal. Its proof is usually carried by an algebraical achievement; but we have got a mode of demonstration different from that.

Let $u_1, u_2, \ldots u_6$ be six points taken on a conic and let v_1, v_2, v_3 be the points where the lines u_1u_6 and u_3u_4, u_1u_2 and u_4u_5, u_2u_3 and u_5u_6 intersect respectively. Then we have to prove the collinearity of v_1, v_2 and v_3; for which purpose it will suffice to assume that the straight lines v_1v_2 and u_2u_3 intersect in v_3' and to prove that v_3' is coincident with v_3.

Through nine given points it is in general possible to determine a cubic curve. Hence there will be such a curve through the nine points $u_1, u_2, \ldots u_6$ and v_1, v_2, v_3'. Let its equation be expressed in the form

$$y^2 = 4x^3 - g_2x - g_3,$$

a form to which a cubic is always liable to be reduced.

Now assume x expressed by an elliptic function of the form $x = \wp(u, g_2, g_3) = \wp u$. Then y will be expressible in terms of a derived function of $\wp u$, that is, by $\wp' u$, so that we have

$$\wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3.$$

Hence from the properties of the functions $\wp u$ and $\wp' u$ we derive the following propositions:

1. *The sum of the parameters u of three collinear points on this curve will be equal to $2m\omega + 2m'\omega'$, where ω and ω' are the two periods of our elliptic function and where m and m' stand for whole numbers; and vice versa.*

2. *The sum of the parameters u for three points on this curve that belong at the same time to a conic will be equal to $2n\omega + 2n'\omega'$, where n and n' are whole numbers. The converse is also true.*

From these propositions the following relations arise:

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1) Journ. of the Phys. School in Tokyo, Vol. 8, pp. 364—366, October, 1900.

$$(1) \qquad u_1 + u_2 + \cdots + u_6 = 2m\omega + 2m'\omega',$$

$$(2) \qquad v_1 + v_2 + v_3 \qquad = 2n\omega + 2n'\omega',$$

$$(3) \qquad \begin{cases} u_6 + u_1 + v_1 = 2n_1\omega + 2n_1'\omega', \\ u_4 + u_3 + v_1 = 2n_2\omega + 2n_2'\omega', \\ u_1 + u_2 + v_2 = 2n_3\omega + 2n_3'\omega', \\ u_5 + u_4 + v_2 = 2n_4\omega + 2n_4'\omega', \\ u_2 + u_3 + v_3' = 2n_5\omega + 2n_5'\omega'. \end{cases}$$

The sum of (1) and (2) being subtracted by the sum of (3), it remains

$$(4) \qquad u_5 + u_6 + v_3' = 2n_6\omega + 2n_6'\omega',$$

which expresses the converse of the proposition 1. It follows therefore that u_5, u_6, v_3' are collinear, and consequently v_3' coincides with v_3, as we had to prove.

J. MIZUHARA, ON HASEGAWA'S THEOREM.[1])

The theorem we call in this place by Hasegawa's name in this: *If in an isosceles trapezoid $ABCD$ with BC and AD to its smaller and greater bases an ellipse S be inscribed touching its four sides, and four circles a, b, c, d whose radii are r, r', r'', r''' are described so as to touch the ellipse and each to two sides of the trapezoid, then $rr' = r''r'''$.*[2])

Here we propose to prove the falsehood of this theorem.

There are four circles that touch any two sides (for instance AB and AD) and the ellipse contained in the angle. Of these four circles two (a_1 and a_2) touch the ellipse externally and the remaining two (a_3 and a_4) internally. These circles will be classified as A and B.

Taking the point A for origin and AB and AD for axes of coordinates, the equation of the ellipse S takes the form

$$(S) \qquad k^2\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 - 4xy = 0,$$

1) Journ. of the Phys. School in Tokyo, Vol. 15, pp. 577—586, November, 1906.

2) This theorem was published in Uchida's Kokon Sankan, 1832, where it is given as one of the problems set by Hori in a Shintō temple in Yedo in 1830. It has attracted much attention in the times of the old mathematical school as well as in recent years, and some papers were published concerning it in the Journal of the Mathematico-Physical Society in Tokyo.

where a and b represent the distances from the origin to the points of contact of the ellipse with the axes.

The equation to a determinate circle a_0 that touches the axes of coordinates is of the same form as (S), only that it should be put $a = b = a_0$ and $k = k_0$. This equation we call (a_0).

The ratio λ that the radii of a_1, a_2, a_3 and a_4 bear to that of a_0 will be given by the equations

$$k_0^2\left(\frac{k^2}{ab} - 1\right)\lambda^2 - kk_0\left[\frac{kk_0}{a_0}\left(\frac{1}{a} + \frac{1}{b}\right) \pm 2\right]\lambda + k_0^2\left(\frac{k_0^2}{a_0^2} - 1\right) = 0,$$

which we term (A) and (B), where k and k_0 are assumed to have positive signs. The upper and lower signs pertain separately to the circles of the classes A and B. (See Terao's paper, *S. Iwata's theorem proved and extended.*)

We shall first study about the circles of the A class.

Denoting by ω the angle between the axes, the equation to the circle a_0 may also be written in the form

$$(a_0') \qquad r_0^2 = (x - a_0 + r_0 \cot \omega)^2 + (y - r_0 \operatorname{cosec} \omega)^2 \\ + 2\,(x - a_0 \cot \omega)\,(y - r_0 \operatorname{cosec} \omega) \cos \omega,$$

where r_0 stands for the radius of the circle.

The ratios of the coefficients of xy and x^2 in the two equations (a_0) and (a_0') should be equal, so that

$$(1) \qquad \frac{2k_0^2}{a_0^2} - 1 : \frac{k_0^2}{a_0^2} = 2 \cos \omega : 1,$$

that is, $k_0 = a_0/\sin \frac{\omega}{2}$, and it is evidently $r_0 = a_0 \tan \frac{\omega}{2}$ (2).

Again we denote by a', b'; a'', b''; and a''', b''' the like magnitudes as a and b for the angles at B, C and D. We also put $AD = n$, $AB = CD = m$.

By writing m for y and a' for x in the equation to the ellipse we have after transposition

$$a'^2 + a'\left\{2\left(\frac{m}{b} - 1\right)a - \frac{4ma^2}{k^2}\right\} + \left(\frac{m}{b} - 1\right)^2 a^2 = 0,$$

an equation that must have two equal roots for a', so that

$$a' = -\left(\frac{m}{b} - 1\right)a + \frac{2ma^2}{k^2}, \text{ and } a' = \frac{m-b}{b}a \text{ or } \frac{b'a}{b},$$

whence we get

$$(3) \qquad \frac{a}{b} = \frac{a'}{b'} = \frac{a+a'}{m} \text{ and } k^2 = \frac{mab}{m-b} = \frac{mab}{b'}$$

and, as we have defined, $\lambda = \frac{r}{r_0}$ (4).

The equation (A) will be transformed by (1), (2), (3), (4) into

$$(5) \quad r^2 - r\tan\frac{\omega}{2}\left\{m + a + a' + 2\sin\frac{\omega}{2}\cdot\sqrt{ma'}\right\} + ma\sin^2\frac{\omega}{2} = 0.$$

The two values of r being denoted by r_1 and r_2 (and we assume $r_1 < r_2$, so that r_1 and r_2 are the radii of the circles a_1 and a_2 respectively, an assumption that we shall follow throughout in the following), we have

$$(A) \quad \begin{cases} r_1 r_2 = ma\sin^2\frac{\omega}{2} = B, \\ r_1 + r_2 = \tan\frac{\omega}{2}\left\{m + a + a' + 2\sin\frac{\omega}{2}\cdot\sqrt{ma'}\right\} = 2A. \end{cases}$$

Similarly for the radii of the circles b_1, b_2, c_1, c_2; d_1, d_2, we obtain the formulae (B), (C), (D), where r_1, r_2, A, B will be denoted with 1, 2, 3 dashes, and the sine and tangent changed in (B) and (C) to the cosine and cotangent, and a, a' in the crooked brackets replaced by a'', a''' in (C) and (D).

We have to establish the relation $aa' = a''a'''$.

The trapezoid together with its inscribed ellipse projected on a certain plane so as the ellipse should become a circle, the side A_0D_0 that corresponds to AD will also be parallel to the side B_0C_0 corresponding to BC, and the angles A_0OB_0 and C_0OD_0, where O is the centre of the circle, become right angles. The radius of that circle, being ϱ, evidently $\varrho^2 = a_0a_0' = a_0''a_0'''$.

But as A_0D_0 and B_0C_0 are parallel, we have by the theory of projected figures

$$(6) \qquad \frac{a}{a_0} = \frac{a'}{a_0'} = \frac{a''}{a_0''} = \frac{a'''}{a_0'''}, \text{ whence } aa' = a''a'''.$$

Hence the products of the first formulae in (A), (B) and (C), (D) are equal, that is,

$$(6') \qquad r_1r_2 \times r_1'r_2' = r_1''r_2'' \times r_1'''r_2'''.$$

In like manner we shall have a similar relation for the circles of the B class.

Again as has been defined

$$(6'') \qquad n = a + a''' = a' + a'' + 2m\cos\omega.$$

From (6) and (6'') follows

$$a' = \frac{(n-2m\cos\omega)(n-a)}{n}, \quad a'' = \frac{(n-2m\cos\omega)\,a}{n},$$

$$(7) \qquad \therefore \quad a'' + a''' = (a+a') + \frac{2m\cos\omega\,(n-2a)}{n}.$$

In our problem ω is necessarily an acute angle. For the sake of convenience we shall assume a to have a value that lies between 0 and $\frac{n}{2}$, which by no means set any restriction on the meaning of our problem, for in such a case where a happens to have a value greater than $\frac{n}{2}$, we can replace a and a' with a''' and a'' respectively so as to adapt itself to our assumption.

From this assumption we see that, unless $a = \frac{n}{2}$, then $a < a'''$, $a + a' < a'' + a'''$ or $a' - a'' < a''' - a$, and evidently $0 < \cos\frac{\omega}{2} - \sin\frac{\omega}{2}$.

By writing $S = \frac{m + m'' + a}{2}$, by (7) we have

$$(8) \quad (A - \sqrt{B})(A' - \sqrt{B'}) = S^2 - \frac{2Sm\cos\omega\,(n-2a)}{n} + \frac{m^2\cos^2\omega\,(n-2a)^2}{n^2}$$

$$- \sin^2\frac{\omega}{2}\cdot ma' - \cos^2\frac{\omega}{2}\cdot ma + 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\cdot m\sqrt{aa'}$$

$$= S^2 + m\left\{a\cos\omega - n\cos^2\frac{\omega}{2}\right\} + \frac{2m^2 a\cos\omega\sin^2\frac{\omega}{2}}{n} + 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\cdot m\sqrt{aa'},$$

and

$$(9) \quad (A'' - \sqrt{B''})(A''' - \sqrt{B'''}) = S^2 - \sin^2\frac{\omega}{2}\cdot ma'' - \cos^2\frac{\omega}{2}\cdot ma'''$$

$$+ 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\cdot m\sqrt{a''a'''}$$

$$= S^2 - \frac{m\sin^2\frac{\omega}{2}(n-2m\cos\omega)\,a}{n} - \cos^2\frac{\omega}{2}(n-a)\,m$$

$$+ 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\cdot m\sqrt{a''a'''}$$

$$= S^2 - \frac{2m^2 a\cos\omega\sin^2\frac{\omega}{2}}{n} + ma\cos\omega - mn\cos^2\frac{\omega}{2}$$

$$+ 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\cdot m\sqrt{a''a'''}.$$

Thus for all values of a, a', a'', a''' and ω we have $(A - \sqrt{B})(A' - \sqrt{B'}) = (A'' - \sqrt{B''})(A''' - \sqrt{B'''})$, that is

$$(10) \quad (\sqrt{r_2} - \sqrt{r_1})(\sqrt{r_2'} - \sqrt{r_1'}) = (\sqrt{r_2''} - \sqrt{r_1''})(\sqrt{r_2'''} - \sqrt{r_1'''}).$$

Again we get formulae for $(A+\sqrt{B})(A'+\sqrt{B'})$ and $(A''+\sqrt{B''})(A'''+\sqrt{B'''})$, which we call (11) and (12). From these equations compared with (6), we deduce the result: unless $a=\frac{n}{2}$, we should have

$$(13)\qquad (\sqrt{r_2}+\sqrt{r_1})(\sqrt{r_2'}+\sqrt{r_1'})<(\sqrt{r_2''}+\sqrt{r_1''})(\sqrt{r_2'''}+\sqrt{r_1'''}).$$

From (10) and (13), or from formulae that lead to these we get

$$(14)\qquad (r_1+r_2)(r_1'+r_2')<(r_1''+r_2'')(r_1'''+r_2'''),$$

$$(14')\qquad (r_2-r_1)(r_2'-r_1')<(r_2''-r_1'')(r_2'''-r_1'''),$$

whence

$$(\alpha)\qquad r_2r_2'+r_1r_1'<r_2''r_2'''+r_1''r_1'''.$$

Both sides of (α) being squared and the equal $4r_1r_2'r_2r_2'$ and $4r_1''r_1'''r_2''r_2'''$ being subtracted from them, the square roots of the remaining expressions give

$$(\beta)\qquad r_2r_2'-r_1r'_1<r_2''r_2'''-r_1''r_1'''.$$

From these we have $r_2r_2'<r_2''r_2'''$ (15), and (6′) being taken into account, $r_1r_1'<r_1''r_1'''$ (16).

Thus we have

Proposition 1. Hasegawa's theorem holds only when the ellipse touches the upper and lower bases at their ends or at their middle points, in other cases (where the ellipse touches the lower base to left of its middle) the product of the radii of the two circles at the left-hand corners is greater than that for other two circles.

Proposition 2. For other four circles of the same class as in 1, Hasegawa's theorem holds only when the ellipse touches the two bases at their middle points; in other cases the concluding sentence in 1 reverses itself, or in other words, the word 'greater' should be changed into the word 'smaller'.

Next something on the properties of the circles of the B class.

Let A_1, A_1', A_1'', A_1''' be the values for the B class corresponding to the A's in the A class, and let r_3, r_4; r_3', r_4'; r_3'', r_4'' and r_3''', r_4''' be the radii of the circles a_3, a_4; b_3, b_4; c_3, c_4 and d_3, d_4. Then

$$(17)\qquad (A_1+\sqrt{B})(A_1'+\sqrt{B'})=(A_1''+\sqrt{B''})(A_1'''+\sqrt{B'''})$$

or

$$(17')\qquad (\sqrt{r_4}+\sqrt{r_3})(\sqrt{r_4'}+\sqrt{r_3'})=(\sqrt{r_4''}+\sqrt{r_3''})(\sqrt{r_4'''}+\sqrt{r_3'''}),$$

as will be seen from (10);

(18) $+\sqrt{(A_1-\sqrt{B})(A_1'-\sqrt{B'})}=\frac{m+a+a'}{2}-\left(\sin\frac{\omega}{2}\cdot\sqrt{ma'}+\cos\frac{\omega}{2}\cdot\sqrt{ma}\right),$

(19) $+\sqrt{(A_1''-\sqrt{B''})(A_1'''-\sqrt{B'''})}=\frac{m+a''+a'''}{2}$

$$-\left(\cos\frac{\omega}{2}\cdot\sqrt{ma'''}+\sin\frac{\omega}{2}\cdot\sqrt{ma''}\right),$$

whence we obtain after reductions

(19′) $(19)-(18)=\frac{m\cos\omega\,(n-2a)}{n}-(\sqrt{n-a}-\sqrt{a})\cos\frac{\omega}{2}\cdot\sqrt{m}$

$$-\sin\frac{\omega}{2}\cdot\sqrt{\frac{m\,(n-2m\cos\omega)}{n}}$$

(20) $$=\frac{m\,(\sqrt{n-a}-\sqrt{a})\left\{n^2\sin\omega\cdot\sqrt{\frac{n-2m\cos\omega}{n}}+2m\cos^2\omega\cdot\sqrt{a(n-a)}-n(n-m\cos\omega)\right\}}{\cdot n\left\{m\cos\omega\cdot(\sqrt{n-a}-\sqrt{a})+n\left[\cos\frac{\omega}{2}\cdot\sqrt{m}-\sin\frac{\omega}{2}\cdot\sqrt{\frac{m\,(n-2m\cos\omega)}{n}}\right]\right\}}$$

If now $a=\frac{n}{2}$, (18) and (19) are equal by (19′). If $a<\frac{n}{2}$, the third factor in (20) may be proved to be negative, so that $(19)-(18)$ becomes negative. In other words

(21) $$(A_1-\sqrt{B})(A_1'-\sqrt{B_1})>(A''-\sqrt{B''})(A_1'''-\sqrt{B'''}),$$

or

(21′) $$(\sqrt{r_4}-\sqrt{r_3})(\sqrt{r_4'}-\sqrt{r_3'})>(\sqrt{r_4''}-\sqrt{r_3''})(\sqrt{r_4'''}-\sqrt{r_3'''}).$$

From (17) and (21), we deduce

(22) $A'A_1'>A_1''A_1'''$ or $(r_4+r_3)(r_4'+r_3')>(r_4''+r_3'')(r_4'''+r_3''),$

and

(22′) $$(r_4-r_3)(r_4'-r_3')>(r_4''-r_3'')(r_4'''-r_3''').$$

Thus the same manipulations, as in the case of the A class circles, lead us at the results:

(23) (i) Unless $a=\frac{n}{2}$, then $r_4r_4'>r_4''r_4'''$,

(24) (ii) Unless $a=0$ or $\frac{n}{2}$, then $r_3r_3'<r_3''r_3'''$.

It follows therefore

Proposition 3. For the smaller four circles of the B class the concluding sentence of proposition 1 should be so changed that the word 'greater' becomes 'smaller'.

Proposition 4. For the greater four circles of the B class the word 'smaller' in the concluding sentence in 2 should be changed into 'greater'.

H. TERAO, ON THE MEAN ERRORS OF OBSERVATIONS.[1]

Let $x, y, z, \ldots$ be n unknown quantities, and let $a_1, b_1, c_1, \ldots,$ $a_2, b_2, c_2, \ldots,$ etc. be rigorously known quantities, while $l_1, l_2, l_3, \ldots$ are numbers resulting from observations.

Then equal weights being relied upon all the observations, the most probable values of the unknowns that are to be determined from the s equations of condition

$$(1) \qquad a_i x + b_i y + c_i z + \cdots = l_i \qquad (i = 1, 2, \ldots, s)$$

will be given by the solution of the following simultaneous system of n linear equations

$$(2) \qquad \begin{cases} [aa]x + [ab]y + [ac]z + \cdots = [al], \\ [ab]x + [bb]y + [bc]z + \cdots = [bl], \ldots, \end{cases}$$

where

$$[aa] = a_1^2 + a_2^2 + \cdots + a_s^2 = \Sigma a_i^2,$$
$$[ab] = a_1 b_1 + a_2 b_2 + \cdots \qquad = \Sigma a_i b_i, \quad \text{etc.}$$

When the mean error or the probable error of the unknowns contained in (2) is to be found, the mean error of the observations must be first known. But the latter cannot be calculated unless we are given the real values — not the probable values — of the unknowns. In the calculation of the mean value of observations we must therefore necessarily content ourselves with an approximate value. But Gauss' explanation of the process known by his name, that is followed by all writers for the determination of this approximate value, has been too unsatisfactory, as is well acquainted. Bertrand speaks of this point, while he is necessitated to adhere to it, there being found no proper substitute.

Recently we have struck a quite different way and have got a tolerably satisfactory result. One point that is most striking lies in the fact that the same final result as obtained by Gauss is arrived at. Thus the genius of the greatest of all mathematicians appears to have divined the best process that ever exists, if he was little able to make a suitable demonstration for it.

Well, we denote by D the determinant of the system (2), and by $D_a, D_b, D_c, \ldots$ its minors that correspond to the elements on the diagonal line from the left upper corner. For other minors we follow such a notation like D_{ab}, that corresponds to $[ab]$.

1) Journ. of the Phys. School in Tokyo, Vol. 5, pp. 323—327, November, 1896.

We shall consider only the case where D does not vanish. Then since (2) is equivalent to

$$D\cdot x = [al]D_a + [bl]D_{ab} + [cl]D_{ac} + \cdots,$$
$$D\cdot y = [al]D_{ab} + [bl]D_b + [cl]D_{bc} + \cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

if we put

$$(3)\qquad \begin{cases} D\cdot \alpha_i = a_i D_a + b_i D_{ab} + c_i D_{ac} + \cdots, \\ D\cdot \beta_i = a_i D_{ab} + b_i D_b + c_i D_{bc} + \cdots, \quad \text{etc.}, \end{cases}$$

the unknowns in (2) will have the values

$$(4)\qquad x = [\alpha l], \quad y = [\beta l], \ldots$$

The residues obtained by the substitution of these values in (1) being denoted by $\Delta_1, \Delta_2, \Delta_3, \ldots$ and the errors of $x, y, z, \ldots$ being denoted by $\xi, \eta, \iota, \ldots$ respectively, the error of the i^{th} observation will be this:

$$a_i(x+\xi) + b_i(y+\eta) + c_i(z+\iota) + \cdots - l_i,$$

or

$$a_i\xi + b_i\eta + c_i\iota + \cdots + \Delta_i.$$

Hence if m be the mean error of each observation, s being assumed a sufficiently large number, we shall have

$$\begin{aligned} sm^2 &= \Sigma(\Delta_i + a_i\xi + b_i\eta + \cdots)^2 \\ &= [\Delta\Delta] + 2([a\Delta]\xi + [b\Delta]\eta + \cdots) + \Sigma(a_i\xi + b_i\eta + \cdots)^2. \end{aligned}$$

But by definition $\Delta_i = a_i x + b_i x + \cdots - l_i$, so that

$$[a\Delta] = [aa]x + [ab]y + [ac]z + \cdots - [al],$$

which vanishes by (2), and similarly

$$[b\Delta] = [c\Delta] = \cdots = 0.$$

Hence

$$sm^2 = [\Delta\Delta] + \Sigma(a_i\xi + b_i\eta + \cdots)^2.$$

Since the second term of the right-hand side of last expression consists of an aggregate of positive and small quantities, and since $a_i\xi + b_i\eta + \cdots$ is the actual error of $a_i x + b_i y + \cdots$, the mean error of the latter will be most appropriate to be replaced by $(a_i\xi + b_i\eta + \cdots)^2$.

Therefore

$$(5)\qquad sm^2 = [\Delta\Delta] + [rr]$$

gives an approximate value of m.

But by (4)

$$a_i x + b_i y + \cdots = (a_i\alpha_1 + b_i\beta_1 + c_i\gamma_1 + \cdots)l_1 \\ + (a_i\alpha_2 + b_i\beta_2 + c_i\gamma_2 + \cdots)l_2 + \cdots \\ + (a_i\alpha_s + b_i\beta_s + c_i\gamma_s + \cdots)l_s,$$

so by a known theorem we have

$$r_i^2 = [(a_i\alpha_1 + b_i\beta + \cdots)^2 + (a_i\alpha_2 + b_i\beta_2 + \cdots)^2 + \cdots]m^2,$$

or

$$\frac{r_i^2}{m^2} = a_i^2[\alpha\alpha] + b_i^2[\beta\beta] + \cdots \\ + 2(a_ib_i[\alpha\beta] + a_ic_i[\alpha\gamma] + b_ic_i[\beta\gamma] + \cdots).$$

Therefore

$$\frac{[rr]}{m^2} = [aa][\alpha\alpha] + [bb][\beta\beta] + \cdots \\ + 2([ab][\alpha\beta] + [ac][\alpha\gamma] + [bc][\beta\gamma] + \cdots). \tag{6}$$

From the first formula in (3) we have

$$D\cdot[a\alpha] = [ac]D_a + [ab]D_{ab} + \cdots = D,$$
$$D\cdot[b\beta] = [ab]D_a + [bb]D_{ab} + \cdots = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\therefore\ [a\alpha] = 1, \quad [b\alpha] = [c\alpha] = \cdots \qquad = 0.$$

It follows similarly

$$[b\beta] = 1, \quad [a\beta] = [c\beta] = \cdots \qquad = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Accordingly we have

$$\left|\begin{array}{l} D\cdot[\alpha\alpha] = D_a, \quad D\cdot[\alpha\beta] = D_{ab}, \ldots, \\ D\cdot[\beta\beta] = D_b, \quad D\cdot[\beta\gamma] = D_{bc}, \ldots, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. \tag{7}$$

Hence we obtain from (6)

$$\frac{D\cdot[rr]}{m^2} = [aa]D_a + [bb]D_b + [cc]D_c + \cdots \\ + 2([ab]D_{ab} + [ac]D_{ac} + [bc]D_{bc} + \cdots)$$

or

$$= [aa]D_a + [ab]D_{ab} + [ac]D_{ac} + \cdots \\ + [ab]D_{ab} + [bb]D_b + [bc]D_{bc} + \cdots \\ + [ac]D_{ac} + [bc]D_{bc} + [cc]D_c + \cdots + \text{etc.}$$

But as each row in the right is an expanded form of $\varDelta$, we have

$$\frac{1}{m^2}D\cdot[rr] = nD, \quad \text{or} \quad [rr] = nm^2.$$

Therefore by (5)

$$sm^2 = [\varDelta\varDelta] + nm^2, \quad \therefore\ m = \sqrt{\frac{[\varDelta\varDelta]}{s-n}},$$

which is no other than the formula of Gauss.

T. Hayashi's notice on the case $D = 0$.[1]

Let the equations of condition, m in number, be written in the form

$$a_{1r}x_1 + a_{2r}x_2 + \cdots + a_{nr}x_n = M_r,$$

$r = 1, 2, \ldots, m$, where m will of course be far greater in comparison to the number of the unknowns, n.

In this case the most probable values of the unknowns will be determinable from the following equations (*normal equations*):

$$[a_s a_1]x_1 + [a_s a_2]x_2 + \cdots = [a_s M],$$

for $s = 1, 2, \ldots, n$, where

$$[a_i a_k] = [a_k a_i] = a_{i1}a_{k1} + a_{i2}a_{k2} + \cdots + a_{im}a_{km},$$

and

$$[a_j M] = a_{j1}M_1 + a_{j2}M_2 + \cdots + a_{jm}M_m,$$
$$(i, j, k = 1, 2, \ldots, n).$$

When the determinant D of the quantities $[a_i a_k]$ does not vanish, the unknowns are to be determined from the foregoing equations. But when $D = 0$, what will it indicate?

We see that the determinant may be written in the form

$$D = \Sigma \begin{vmatrix} a_{1p} & a_{1q} & a_{1r} & \ldots & a_{1s} \\ a_{2p} & a_{2q} & a_{2r} & \ldots & a_{2s} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{np} & a_{nq} & a_{nr} & \ldots & a_{ns} \end{vmatrix}^2,$$

which will appear from the multiplication of determinants. The symbol of summation extends for the whole domain where every n different characters $p, q, r, \ldots, s$ will be taken from among $1, 2, 3, \ldots, m$. When D is $= 0$, therefore, every term of the aggregate will separately vanish, or

$$\begin{vmatrix} a_{1p} & a_{1q} & \ldots \\ a_{2p} & a_{2q} & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

for all possible values of $p, q, \ldots$

1) Journ. of Phys. School in Tokyo, Vol. 6, pp. 117—119, April, 1897.

But $p, q, r, \ldots$ are n different characters out of $1, 2, \ldots, m$. The m equations of condition constructed by observations are therefore such that any n out of them are not sufficient to give the values of the unknowns. Therefore the condition $D=0$ shows that all the observations result in an inefficiency, so that nothing can be learnt of the values of the unknowns; and consequently that the mean error of observations cannot be found in this case.

H. TERAO, ON THE MOTION OF A PLANE ON A PLANE.[1])

About the motion of a plane Q on another plane P we base all our considerations on the following theorem:

The motion of Q at every moment of time is a revolution around a point called the centre of momentary revolution.

Hence we know that the normal of the trajectory of any point in Q will pass through the centre of momentary revolution. If we know therefore the trajectories of any two points, the centre of momentary revolution at any moment will be found by the intersection of their normals, and consequently the normal to the trajectory of any other point can be constructed.

Now we imagine the motion of P in relation to Q, when Q moves on P. If we denote by $B_1, B_2, \ldots$ respectively, the trajectories described on Q by the points $A_1, A_2, \ldots$, that belong to P, these curves B on Q will always pass through the respective points A, when Q is conceived to be moving on P. The relative velocities of the points of P for the movement of P in relation to Q will have the opposite directions to those of the points of Q in the motion of Q in relation to P; so that these two motions will be at every moment oppositely directed revolutions around the same point. Hence at every moment the normals of the B's at the points A's will all meet in the centre of momentary revolution. Therefore *one of these curves B's may be replaced for the trajectory of a point in Q for the purpose to find this centre.*

Application 1. M and M' are points on the radius vector ON drawn from a fixed point O to any point N on a given curve C, such that their distances from N are equal to given lengths. The normals drawn at M and M' to the loci described by these points when the radius vector revolves along the curve are required to be constructed.

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1) Journ. of Phys. School in Tokyo, Vol. 6, pp. 141—142, May, 1897.

These loci may be considered as the trajectories of M and M', when a plane Q so moves on a given plane P that the straight line MM' on Q always passes through C and so that a point N on it has C for its trajectory. The centre of momentary revolution of Q is on the normal of C at N; it is also on the perpendicular of ON at O. If I is the intersection of these two straight lines, then MI and $M'I$ are the normals required.

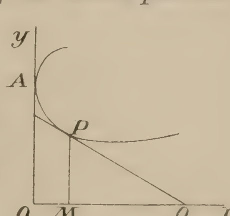
When the curve C is a straight line or a circle that passes through O, the loci will be conchoids or rimaçons.

Application 2. To draw a normal to a rimaçon.

If two orthogonally intersecting straight lines on a moving plane Q pass through the fixed points O and O' respectively, their point of intersection N will naturally describe the circle with OO' as diameter. Hence if the point I, the intersection of ON and $O'N'$, perpendiculars erected at O and O' respectively, be joined to the points M and M', we obtain the normals drawn at M and M' to the rimaçons, which are the loci of the points M and M'.

T. HAYASHI, ON A KINEMATICAL PROBLEM.[1])

Of two moving particles the one moves along a straight line and with a uniform velocity, while the other moves with a uniform velocity towards the first particle; what will be the trajectory of the second particle?



This is a well-known problem in kinematics.

Let Ox be the path of the first particle and v its velocity, and AP the path of the second particle and u its velocity. If the two particles are at Q and P respectively, the straight line PQ will touch to the trajectory of the second particle.

Let AO be the position of PQ when it is perpendicular to Ox. Then we have $\frac{AP}{u} = \frac{OQ}{v}$.

If we measure the length of the curve from A and put $AP = s$, then writing $e = \frac{v}{u}$, we have

$$es = x - y\frac{dx}{dy},$$

whence we obtain the equation to the curve

1) Journ. of Phys. School in Tokyo, Vol. 14, pp. 203—207, May, 1905.

$$2\left(x - \frac{ae}{1-e^2}\right) = \frac{a^{-e}y^{1+e}}{1+e} - \frac{a^e y^{1-e}}{1-e} \qquad (e \neq 1),$$

and

$$2\left(x + \frac{a}{4}\right) \quad = \frac{y^2}{2a} - a \log \frac{y}{a} \qquad (e = 1),$$

where a denotes the length of OA.

When the value of e is ≥ 1, the curve does not meet the x-axis, which will be approached in an asymptotic way. We have to say nothing about this case.

The curve we have to examine is that for which e is < 1.

In this case the curve meets the axis of x and has a contact with it. This point of contact is given in ordinary works as a point of inflexion of the curve. But it is *not necessarily a point of inflexion;* and consequently for $x = \frac{ae}{1-e^2}$ there do not necessarily exist the two values of y

$$y = \pm a\left(\frac{1+e}{1-e}\right)^{\frac{1}{2e}},$$

as Tait and Steel state in their *Dynamics of a Particle.*

The point of contact is evidently $\left(\frac{ae}{1-e^2}, 0\right)$, and the two particles meet in this point, so that their distance here vanishes. For we have

$$PQ = \sqrt{\{(OQ - OM)^2 - PM^2\}} = \pm\, y\frac{ds}{dy}$$

$$= \pm\, \frac{y}{2}\left\{\left(\frac{y}{a}\right)^e + \left(\frac{a}{y}\right)^e\right\} = \pm\, \frac{y^{2e} + a^{2e}}{2a^e} \cdot y^{1+e} \qquad (e < 1).$$

This point will be termed X.

By successive differentiations we get

$$\frac{1}{2}\frac{dy}{dx} = \frac{a^e y^e}{y^{2e} - a^{2e}}, \quad -\frac{1}{4a^{2e}}\frac{d^2y}{dx^2} = \frac{y^{2e-1}(y^{2e}+a^{2e})}{(y^{2e}-a^{2e})^3},$$

$$-\frac{1}{4a^{2e}}\frac{d^3y}{dx^3} = \frac{y^{2e-2}}{(y^{2e}-a^{2e})^4}\left[(y^{2e}-a^{2e})\{(4e-1)y^{2e}\right.$$
$$\left. + (2e-1)a^{2e}\} - y^{2e}(y^{2e}+a^{2e})6e\right], \quad \text{etc.},$$

so that we have

1°. For $1 > e > \frac{1}{2}$, at the point X

$$\frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = \infty.$$

2°. For $e = \frac{1}{2}$,

$$\frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} = \frac{4}{a}, \quad \frac{d^3y}{dx^3} = \frac{16}{a^2}.$$

3^0. For $e < \frac{1}{2}$,

$$\frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} = \infty, \quad \frac{d^3y}{dx^3} = \infty.$$

If we put $e = \frac{p}{q}$, where p and q are positive integers that have no common factor, then $1 \pm e = \frac{p \pm q}{p}$ are reduced fractions. And from

$$x = \frac{ae}{1-e^2} + \frac{1}{2}\left(\frac{a^{-e}y^{1+e}}{1+e} - \frac{a^e y^{1-e}}{1-e}\right),$$

the values of x can be made out that correspond to a value of y.

1^0. From the values $y' = 0$, $y'' = 0$, $y''' \neq 0$ we cannot conclude at once that the point X is *a point of inflexion*. We can only say that *the radius of curvature* at this point is *infinitely great*. From the equation to the trajectory we see that

(i) y cannot be negative if p be even. Two values of x correspond to a positive value of y. The curve has a node at

$$\left(\frac{ae}{1-e}, \quad \pm\left(\frac{1+e}{1-e}\right)^{\frac{1}{2e}} a\right).$$

Thomson and Tait have committed a fault when they refer to X as a point of inflexion for $e = \frac{3}{4}$. (See *Thomson and Tait*, *Natural Philosophy, Vol. I.*) O. Arakawa also falls into the same error in his *Sūgaku Yōgi*.

(ii) If p be odd, y can become negative. A value of x corresponds to a value of y. In this case, for $x = \frac{ae}{1-e^2}$, there are two points

$$y = \pm\left(\frac{1+e}{1-e}\right)^{\frac{1}{2e}} a.$$

(a) When p is odd and q even, X is *a point of inflexion*.

(b) When p is odd and q odd, X is *a cusp*.

2^0. When e has the value $\frac{1}{2}$, the equation of the curve is

$$9a\left(x - \frac{2a}{3}\right) = y(y - 3a)^2,$$

which represents a cubical parabola and y must be always positive. At the point X we have $y' = 0$ and $y'' = \frac{4}{a}$, so that the radius of curvature is $\frac{a}{4}$.

3^0. The case for $e < \frac{1}{2}$. The same as in the case 1^0.

In the case *when the point X is a cusp,* very queer will be observed the behavior of the moving particle. But if we look on the way in which the differential equation has been constructed, it will all become clear. *The first particle comes from O to X and then it should be considered to reverse its direction on a sudden and move back towards O.*

INDEX OF PROPER NAMES.[1]

1) The names of books are printed in italics. Chinese and older Japanese books only are referred to.

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